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THE BOUNDARY PROBLEMS AND DEVELOPMENTS ASSOCIATED WITH A SYSTEM OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

By George D. Birkhoff and Rudolph E. Langer.[1]

Introduction.

It is the purpose of this paper to develop in outline the theory of a system of n ordinary linear differential equations of the first order containing a parameter and subject to certain boundary conditions. Toward this end the notation of matrices is used. For the convenience of the reader the paper opens with a brief review of the fundamentals of matrix algebra and the integration and differentiation of matrices. This is followed by an expository discussion of the homogeneous and non-homogeneous differential matrix equations of the first order. The major portion of the treatment is devoted, however, to the homogeneous differential vector equation with a complex parameter in its coefficient, and to the system composed of such an equation and suitable boundary conditions. The solutions of the equation for large values of the parameter are discussed and the formal development of a vector of arbitrary functions into a series of solutions of the system is obtained. The paper closes with the proof of the convergence of this development under appropriate conditions, which, in the ordinary notation, establishes the possibility of simultaneously expanding n arbitrary functions in terms of the characteristic solutions of a properly restricted differential system of the type

$$y_i'(x) = \sum_{k=1}^{n} \{a_{ik}(x)\lambda + b_{ik}(x)\} y_k(x),$$

$$\sum_{k=1}^{n} \{\alpha_{ik} y_k(a) + \beta_{ik} y_k(b)\} = 0, \quad i = 1, 2, \ldots n.$$[2]

When reduced to a single equation of the nth order this includes as a special case the expansions obtained by Birkhoff in 1908.

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[1] Much of the material preceding the proof of convergence is due to Birkhoff, having been developed by him in lectures at Harvard University in the fall of 1920. The reorganization of this material into its present form, the treatment of the irregular case, and the proof of convergence are due to Langer.

[2] For other developments in this field and more complete references see the following papers in the Transactions of the American Mathematical Society: Birkhoff, *On the Asymptotic Character of the Solutions of Certain Linear*

Section I.

Definitions.[3]

An array of elements of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ a_{31} & \ldots & \ldots\ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{pmatrix}$$

in which the number of rows equals the number of columns is called a square matrix, and is denoted either by (a_{ij}) or by A. Two such matrices are said to be equal when, and only when, every element of the one is equal to the correspondingly situated element of the other.

The sum of two matrices, (a_{ij}) and (b_{ij}), having the same number of rows and columns, is by definition the matrix $(a_{ij} + b_{ij})$, from which it follows that matrix addition is both commutative and associative, i.e., $A + B = B + A$, and $A + (B + C) = (A + B) + C$.

The product of two n rowed matrices A and B is defined by the identical equation

$$(a_{ij})\,(b_{ij}) \equiv \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right),$$

on the basis of which it is easily verified that matrix multiplication is both associative and distributive, i.e.

$$A(BC) = (AB)C,$$
$$A(B + C) = AB + AC.$$

Differential Equations Containing a Parameter, and *Boundary Value and Expansion Problems of Ordinary Linear Differential Equations*, vol. 9 (1908), p. 219 and p. 373.

Wilder, *Expansion Problems of Ordinary Linear Differential Equations with Auxiliary Conditions at More than Two Points*, vol. 18 (1917), p. 415.

Hopkins, *Some Convergent Developments Associated with Irregular Boundary Conditions*, vol. 20 (1919), p. 245.

Hurwitz, *An Expansion Theorem for a System of Linear Differential Equations of the First Order* (about to appear in vol. 22 (1921)).

Langer, *Developments Associated with a Boundary Problem not Linear in the Parameter* (about to appear in vol. 22 (1921)).

[3] For a more ample discussion of the theory of matrices see Bôcher, M., *Introduction to Higher Algebra*. New York; The Macmillan Co., 1907.

That it is in general not commutative is a consequence of the fact that $\sum_{k=1}^{n} a_{ik}b_{kj}$ is in general not equal to $\sum_{k=1}^{n} b_{ik}a_{kj}$. The rearrangement of the factors in a matrix product is, therefore, as a rule, not permissible.

The multiplication of a scalar into a matrix has simply the effect of multiplying each element of the matrix by the scalar. Thus if k is a scalar, then $kA = Ak = (ka_{ij})$. Conversely, any factor common to all the elements of a matrix can be factored from the matrix.

Two special matrices must be mentioned, namely $O = (0)$, the *zero matrix,* and $I = (\delta_{ij})$, the *unit matrix,* where $\delta_{ij} = 0$ when $i \neq j$, $\delta_{ii} = 1$. These matrices satisfy respectively the relations

$$AO = OA = O$$

and

$$AI = IA = A.$$

The determinant formed from the elements of a matrix without changing the order of the array is called the *determinant of the matrix.* The alternative notations $| a_{ij} |$ or $| A |$ for the determinant of the matrix (a_{ij}) will be used.

Given a matrix A it follows that if $| A | \neq 0$ then there exists a unique solution in the x's for each of the linear systems

$$\sum_{k=1}^{n} a_{ik}\, x_{kj_0} = \delta_{ij_0}, \qquad i = 1,\ 2,\ \ldots n,$$

where j_0 ranges from 1 to n. This means that there exists a unique set of n^2 quantities x_{ij} such that

$$\sum_{k=1}^{n} a_{ik}\, x_{kj} = \delta_{ij}, \quad i, j, = 1, 2, \ldots \mathrm{n},$$

i.e. there exists a unique matrix (x_{ij}) such that

$$A \cdot (x_{ij}) = I.$$

This matrix (x_{ij}) is denoted by the symbol A^{-1} and is called the *inverse of* A. From its derivation it is seen to satisfy the relation $AA^{-1} = I$.

Either of the relations, $AX = O$ or $XA = O$, leads, on the assumption that $| A | \neq 0$, to the conclusion $X = O$, as is evident from the theory of the systems of linear equations to which the matrix equations

are equivalent. It follows from the relations $AA^{-1} = I$ and $IA = AI$, however, that $AA^{-1}A = IA = AI$, i.e., $A(A^{-1}A - I) = O$. Hence if A is any matrix for which $|A| \neq 0$, it is seen that

$$A^{-1}A - I = O, \text{ i.e., } A^{-1}A = I = AA^{-1}.$$

In accordance with the following definitions, namely

$$\int_\alpha^\beta A(x)\,dx = \left(\int_\alpha^\beta a_{ij}(x)\,dx\right)$$

and

$$\frac{dA(x)}{dx} = \left(\frac{da_{ij}(x)}{dx}\right),$$

a matrix is seen to be integrable or differentiable if and only if this is true of each of its elements. It is also clear that if C is a matrix of constants, then $\frac{dC}{dx} = O$, while for any product

$$\frac{d}{dx}AB = A\frac{dB}{dx} + \frac{dA}{dx}B.$$

Section II.

The equation $Y'(x) = A(x)\,Y(x)$.[4]

Consider any matrix of functions $Y(x)$ which satisfies (i.e. is a solution of) the equation

$$(1) \qquad Y' = AY.$$

In accordance with the rules for determinants we have

$$\frac{d|Y|}{dx} = \begin{vmatrix} y'_{11} & \dots & y'_{1n} \\ y_{21} & \dots & y_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ y_{n1} & \dots & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & \dots & y_{1n} \\ y'_{21} & \dots & y'_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ y_{n1} & \dots & y_{nn} \end{vmatrix} + \dots + \begin{vmatrix} y_{11} & \dots & y_{1n} \\ y_{21} & \dots & y_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ y'_{n1} & \dots & y'_{nn} \end{vmatrix},$$

[4] For the general theory of matrix differential equations see Schlesinger, L., *Vorlesungen über lineare Differentialgleichungen.* Leipzig; B. G. Teubner, 1908.

whence, substituting from the system

$$y'_{ij} = \sum_{k=1}^{n} a_{ik}\, y_{kj}, \quad i, j, = 1, 2, \ldots n$$

(which is equivalent to equation (1)) we obtain

$$\frac{d\,|\,Y\,|}{dx} = \begin{vmatrix} \sum_k a_{1k}\, y_{k1} \ldots \sum_k a_{1k}\, y_{kn} \\ y_{21} \quad \ldots\ldots\ldots\ldots \quad y_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n1} \quad \ldots\ldots\ldots\ldots \quad y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} \quad \ldots\ldots\ldots\ldots \quad y_{1n} \\ \sum_k a_{2k}\, y_{k1} \ldots \sum_k a_{2k}\, y_{kn} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n1} \quad \ldots\ldots\ldots\ldots \quad y_{nn} \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} y_{11} \quad \ldots\ldots\ldots\ldots \quad y_{1n} \\ y_{21} \quad \ldots\ldots\ldots\ldots \quad y_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \sum_k a_{nk}\, y_{k1} \ldots \sum_k a_{nk}\, y_{kn} \end{vmatrix},$$

i.e. $\dfrac{d}{dx}\,|\,Y\,| = a_{11}\,|\,Y\,| + a_{22}\,|\,Y\,| + \ldots + a_{nn}\,|\,Y\,|.$

Let us suppose, now (i), that the elements of $A(x)$, $Y(x)$, and $Y'(x)$ are each continuous in an interval $a \leqq x \leqq b$, and (ii), that at some point x of this interval $|\,Y\,| \neq 0$. Then throughout a neighborhood of the point in question

$$\frac{d\,|\,Y\,|}{|\,Y\,|} = \sum_{k-1}^{n} a_{kk}\, dx,$$

or, integrating,

$$|\,Y\,| = ce^{\int \Sigma\, a_{kk}\, dx}.$$

Inasmuch as the right-hand side of this equation cannot vanish if $c \neq 0$, it is seen that the hypothesis that $|\,Y\,| \neq 0$ for some x leads to the conclusion that $|\,Y\,|$ differs from zero for all x. Thus we infer the

THEOREM: If $Y(x)$ is a matrix of functions which satisfies equation (1), while $|\,Y(x_0)\,| = 0$, $a \leqq x_0 \leqq b$, then $|\,Y(x)\,| \equiv 0$, $a \leqq x \leqq b$.

A matrix of functions $Y(x)$ of this type which satisfies equation (1)

and whose determinant $| Y(x) | \neq 0$, is called a *matrix solution* of equation (1). We assume at least one such solution to exist.

THEOREM: If $Y(x)$ is any (matrix) solution of equation (1), and C is any matrix of constants (for which $| C | \neq 0$), then $\bar{Y}(x)$, defined by $\bar{Y}(x) = Y(x)C$, is also a (matrix) solution of equation (1).

Proof: We have
$$\frac{d\bar{Y}}{dx} = \frac{d}{dx} YC$$
$$= \frac{dY}{dx} C + Y \frac{dC}{dx}$$
$$= \frac{dY}{dx} C.$$

But by hypothesis
$$\frac{dY}{dx} = AY.$$

Hence
$$\frac{d\bar{Y}}{dx} = AYC,$$

i.e.
$$\frac{d\bar{Y}}{dx} = A\bar{Y}.$$

Moreover, since the product of two matrices is derived in the same manner as that of two determinants, it follows that $| \bar{Y} | = | Y | | C |$. Hence $| \bar{Y} | \neq 0$, if $| C | \neq 0$ and $| Y | \neq 0$. Q.E.D.

THEOREM: Given any matrix of constants Y_0 for which $| Y_0 | \neq 0$, then there exists a matrix solution of equation (1) which for any preassigned x, say $x = x_0$, reduces to the matrix Y_0.

Proof: It has already been shown that when $Y(x)$ is a matrix solution of equation (1), then $Y(x)C$ is also such a solution, where C may be any matrix of constants whose determinant is not zero. Then in particular C may be chosen as the matrix $Y^{-1}(x_0)\, Y_0$, whereupon it follows that $Y(x)\, Y^{-1}(x_0)\, Y_0$ is also a matrix solution of equation (1). This solution, however, obviously reduces to the matrix Y_0 when $x = x_0$. Q. E. D.

THEOREM: If $Y(x)$ is a matrix solution of equation (1) then $Y(x)C$ is the most general solution of equation (1).

Proof: Let $\overline{Y}(x)$ be any solution of equation (1) whatever. Then the relation $\overline{Y}(x) = Y(x)\Phi(x)$ defines a matrix $\Phi(x)$, and substituting from this expression in equation (1) we have

$$\frac{d}{dx} Y\Phi = AY\Phi,$$

i.e.
$$Y\frac{d\Phi}{dx} + \frac{dY}{dx}\Phi = AY\Phi.$$

But by hypothesis
$$\frac{dY}{dx} = AY.$$

Hence
$$Y\frac{d\Phi}{dx} + AY\Phi = AY\Phi,$$

i.e.
$$Y\frac{d\Phi}{dx} = O.$$

Inasmuch as $| Y(x) | \neq 0$, it follows that $\dfrac{d\Phi}{dx} = O$, and $\Phi(x) \equiv C$.

Q. E. D.

A mere interchange of the rôles played by the rows and the columns of the matrices involved transforms the discussion carried out thus far for equation (1) to the corresponding discussion for the equation $Y' = YA$. The facts in the two cases may, therefore, be established by precisely the same methods.

The pair of related equations

$$(1) \qquad Y' = AY,$$

$$(2) \qquad Z' = -ZA,$$

are said to be *adjoint*, either being the adjoint of the other.

THEOREM: If $Y(x)$ is any matrix solution of equation (1), and $Z(x)$ is any matrix solution of equation (2), then

$$Z(x)\, Y(x) \equiv C.$$

Proof: From the relation

$$\frac{d}{dx} ZY = \frac{dZ}{dx} Y + Z\frac{dY}{dx}$$

it follows, upon substituting from equations (1) and (2), that

$$\frac{d}{dx} ZY = (-ZA)\, Y + Z(AY).$$

Hence $\qquad \dfrac{d}{dx} ZY = 0,$ and $Z(x)\, Y(x) \equiv C.$ Q. E. D.

CONVERSE THEOREM: If $Y(x)$ is any matrix solution of equation (1), and the matrix $Z(x)$ is defined by the relation $Z(x)\, Y(x) \equiv C$, where $|\, C\, | \neq 0$, then $Z(x)$ is a matrix solution of equation (2).

Proof: We have

$$\frac{d}{dx} ZY = 0,$$

i.e.

$$\frac{dZ}{dx} Y + Z \frac{dY}{dx} = 0.$$

In virtue of equation (1), therefore,

$$\frac{dZ}{dx} Y + ZAY = 0,$$

and it follows, since $|\, Y\, | \neq 0$, that

$$\frac{dZ}{dx} = -ZA.$$

Moreover, $|\, Z\, | = |Y^{-1}| \cdot |\, C\, | \neq 0.$ Q. E. D.

Any pair of solutions $Y(x)$ and $Z(x)$ of equations (1) and (2) respectively which satisfy the relation $Z(x)\, Y(x) \equiv I$, are said to be *associated solutions*. Thus if $Y(x)$ is any matrix solution of equation (1) the associated matrix solution of equation (2) is $Z(x) = Y^{-1}(x)$.

A differential equation is said to be *self-adjoint* when and only when it is identical with its adjoint after interchange of rows and columns. A necessary and sufficient condition that the equation (1) be self-adjoint is readily seen to be that $a_{ij} = -a_{ji}$ for all i and j.

SECTION III.

The equation $Y'(x) = A(x)Y(x) + B(x)$.

THE EXISTENCE THEOREM: Given the equation

$$Y' = AY + B, \tag{3}$$

where $A(x)$ and $B(x)$ are matrices of continuous functions, $a \leqq x \leqq b$, then there exists a unique matrix $Y(x)$ whose elements are functions which are continuous together with their first derivatives, $a \leqq x \leqq b$, and which satisfies equation (3) as well as the condition $Y(x_0) = Y_0$, the matrix Y_0 being any prescribed matrix of constants.

Proof: By means of the relation

$$Y_m(x) = Y_0 + \int_{x_0}^{x} \{A(t)\, Y_{m-1}(t) + B(t)\} dt$$

it is possible to define the following infinite sequence of matrices $Y_1(x), Y_2(x), \ldots Y_m(x), \ldots$ which satisfy the relations

$$\begin{array}{ll} Y_1'(x) = A(x)\, Y_0 + B(x), & Y_1(x_0) = Y_0, \\ Y_2'(x) = A(x)\, Y_1(x) + B(x), & Y_2(x_0) = Y_0, \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ Y_m'(x) = A(x)\, Y_{m-1}(x) + B(x), & Y_m(x_0) = Y_0. \\ \cdots\cdots\cdots\cdots & \cdots\cdots \end{array}$$

Then setting $\qquad Y_m(x) - Y_{m-1}(x) = U_m(x)$

in the identical equation

$$Y_m(x) \equiv Y_0 + (Y_1(x) - Y_0) + (Y_2(x) - Y_1(x)) + \ldots + (Y_m(x) - Y_{m-1}(x))$$

we have

$$Y_m(x) \equiv Y_0 + U_1(x) + U_2(x) + \ldots + U_m(x).$$

Moreover, $\qquad U_m(x_0) = O,$

while
$$\begin{aligned} U_m'(x) &= Y_m'(x) - Y_{m-1}'(x) \\ &= A(x)\, \{Y_{m-1}(x) - Y_{m-2}(x)\} \\ &= A(x)\, U_{m-1}(x)\,. \end{aligned}$$

Denoting by the symbol $\ll$ the fact that each element of the matrix of constants on the right represents a value as large as the largest numerical maximum attained by any element of the matrix on the left, we have, since $A(x)$ and $U_1(x)$ are continuous in the closed interval $a \leqq x \leqq b$,

$$U_1(x) \ll (k),$$
$$A\ (x) \ll (\alpha)\ .$$

From this it follows that

$$U_2'(x) = A(x)\ U_1\ (x) = \left(\sum_{l=1}^{n} a_{il}\, u_{lj}^{(1)}\right) \ll (n\alpha k)$$

and integrating

$$U_2(x) \ll (kn\alpha \mid x - x_0 \mid).$$

Likewise $\quad U_3' = AU_2 = \left(\sum_{l=1}^{n} a_{il}\, u_{lj}^{(2)}\right) \ll \left(\sum_{l=1}^{n} \alpha \cdot kn\alpha \mid x - x_0 \mid\right),$

$$U_3(x) \ll \left(kn^2\alpha^2\ \frac{\mid x - x_0 \mid^2}{2!}\right),$$

and similarly for $l = 3, 4, \ldots,$

$$U_{l+1}(x) \ll \left(kn^l\alpha^l\ \frac{\mid x - x\cdot \mid^l}{l!}\right) \ll k \left(\frac{\{na\overline{b-a}\}^l}{l!}\right).$$

However the infinite series of matrices

$$\left(\sum_{l=0}^{\infty} \frac{\{an\overline{b-a}\}^l}{l!}\right)$$

converges to the matrix

$$k\ (e^{an\overline{b-a}} - 1).$$

Hence the series

$$Y_0 + \sum_{l=1}^{\infty} U_l(x)$$

converges uniformly throughout the interval $a \leqq x \leqq b$, and since its terms are matrices of continuous functions the matrix $Y(x)$ which it

represents is necessarily one with elements continuous for $a \leqq x \leqq b$. Now

$$Y_m(x) = Y_0 + \int_{x_0}^{x} \{A(t) Y_{m-1}(t) + B(t)\}\, dt$$

by definition, and since the convergence of $Y_m(x)$ to $Y(x)$ is uniform

$$\lim_{m=\infty} Y_m(x) = Y_0 + \int_{x_0}^{x} \lim_{m=\infty} \{ A(t)\, Y_{m-1}(t) + B(t)\}\, dt,$$

namely,
$$Y(x) = Y_0 + \int_{x_0}^{x} \{A(t) Y(t) + B(t)\}\, dt.$$

Differentiating we see that the elements of $Y'(x)$ are continuous, $a \leqq x \leqq b$, while $Y'(x) = A(x)\, Y(x) + B(x)$. Hence the solution exists as stated.

To prove the solution unique suppose that both $Y(x)$ and $\overline{Y}(x)$ are solutions of equation (3) each reducing to the matrix Y_0 for $x = x_0$, i.e.

$$Y'(x) = A(x)\, Y(x) + B(x), \qquad Y(x_0) = Y_0,$$
$$\overline{Y}'(x) = A(x)\, \overline{Y}(x) + B(x), \qquad \overline{Y}(x_0) = Y_0.$$

Now $Y(x) - \overline{Y}(x) \equiv D(x)$ is a matrix whose elements are continuous, $a \leqq x \leqq b$, and which satisfies the relations

$$D'(x) = A(x)\, D(x), \quad D(x_0) = O.$$

Let any neighborhood of the point x_0 be chosen, say $| x - x_0 | \leqq \delta$. Then, if d denotes the largest numerical maximum of any element of $D(x)$ for any x of this neighborhood so that $D(x) \ll (d)$,

$$A(x)\, D(x) \ll (n\alpha d),$$

and
$$D'(x) \ll (n\alpha d).$$

Hence
$$D(x) \ll (n\alpha d\, | x - x_0 |\,) \ll (n\alpha d\delta).$$

But for some x of the interval at least one element of $D(x)$ is numerically equal to d (by the definition of d), and for this element it follows that $d \leqq n\alpha d\delta$, i.e. $d\, \{ 1 - n\alpha\delta\} \leqq 0$.

However, since α and n are fixed numbers and δ can be chosen so that $\delta < \dfrac{1}{n\alpha}$ this leads to a contradiction unless $d = 0$. It follows that there always exists a finite interval throughout which $d = 0$ and hence throughout which $D(x) \equiv 0$. Inasmuch as this implies that $D(x) \equiv 0$, $a \leqq x \leqq b$, the solutions $Y(x)$ and $\overline{Y}(x)$ must be identically the same. Q. E. D.

The solution of equation (3) can be easily expressed in terms of the solutions $Y_h(x)$ and $Z_h(x)$ of the homogeneous equations (1) and (2) respectively. Thus, multiplying both sides of equation (3) on the left by $Z_h(x)$ we have

$$Z_h \frac{dY}{dx} = Z_h A Y + Z_h B,$$

which, in view of equation (2), can be written

$$Z_h \frac{dY}{dx} = -\frac{dZ_h}{dx} Y + Z_n B,$$

i.e.

$$\frac{d}{dx} Z_h Y = Z_h B.$$

Integrating we obtain

$$Z_h Y = C + \int_a^x Z_h(t)\, B(t)\, dt,$$

while the multiplication of this equation on the left by $Y_h(x)$, the solution associated with $Z_h(x)$, yields

$$Y(x) = Y_h(x)\, C + \int_a^x Y_h(x)\, Z_h(t)\, B(t)\, dt. \tag{4}$$

This is the general solution of equation (3). In consequence any particular solution $\overline{Y}(x)$ may be written

$$\overline{Y}(x) = Y_h(x)\, \overline{C} + \int_a^x Y_h(x)\, Z_h(t)\, B(t)\, dt.$$

Subtracting this from equation (4), however, we obtain

$$Y(x) = \overline{Y}(x) + Y_h(x) \{ C - \overline{C} \},$$

from which it is seen that if $\overline{Y}(x)$ is any particular solution of equation (3) the general solution is given by

$$(5) \qquad Y(x) = \overline{Y}(x) + Y_h(x)\, C.$$

If in particular C is taken as $C = O$ in formula (4), the solution which is characterized by the fact that $Y(\alpha) = O$ is obtained. This is called the *principal solution* at $x = \alpha$. Since the choice of α as a limit of integration is unrestricted, the expression for the principal solution at any chosen point is at hand.

Section IV.

The differential system

$$Y'(x)\cdot = A(x)Y(x)\cdot + B(x)\cdot$$
$$W_aY(a)\cdot + W_bY(b)\cdot = O.$$

A matrix in which any row (or column) is precisely like every other row (or column) is called a *vector*. That a particular matrix is a vector is indicated by means of a dot suitably placed in relation to the letter designating the matrix in question, the dot preceding in case it is a vector in which the rows are the same and succeeding in case it is a vector of identical columns.

Thus $$\cdot A = (a_j) = \begin{Vmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \end{Vmatrix},$$

and $$A\cdot = (a_i) = \begin{Vmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_n & a_n & \dots & a_n \end{Vmatrix}.$$

That $\cdot AB = \cdot C$, and that $AB\cdot = C\cdot$, are facts readily established by direct reference to the rules for multiplication. A matrix (a) all of whose elements are the same is written $\cdot A\cdot$. From the preceding statement it is clear that $\cdot AB\cdot = \cdot C\cdot$.

Given W_a and W_b, any two constant matrices for which $|W_a| \neq 0$ and $|W_b| \neq 0$, then the two corresponding homogeneous vector differential systems

$$(6) \qquad \begin{cases} Y'(x)\cdot = A(x)\, Y(x)\cdot \\ W_aY(a)\cdot + W_bY(b)\cdot = 0 \end{cases}$$

and

$$(7) \qquad \begin{cases} \cdot Z'(x) = - \cdot Z(x)\, A(x) \\ \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0 \end{cases}$$

are said to be *adjoint.*

THEOREM: The number of linearly independent solutions of system (6) is always equal to the number of linearly independent solutions of system (7).

Proof: It has been shown that if $\overline{Y}(x)$ is any matrix solution of the differential equation (1), the most general solution is $Y(x) = \overline{Y}(x)C$. From this, $C = \overline{Y}^{-1}(x)\, Y(x)$, and if the solution $Y(x)$ is a vector $Y(x)\cdot$ then C will be a vector $C\cdot$.

The general solution of the differential equation in system (6) is, therefore,

$$Y(x)\cdot = \overline{Y}(x)\, C\cdot,$$

and since the substitution of this in the boundary conditions gives

$$(8) \qquad W_a\, \overline{Y}(a)\, C\cdot + W_b\, \overline{Y}(b)\, C\cdot = 0,$$

$Y(x)\cdot$ is seen to be a solution of system (6) when and only when $C\cdot$ is a solution of the equation (8). Moreover, it is readily seen that a necessary and sufficient condition that a set of solutions of system (6) be linearly independent is that the corresponding solutions of equation (8) be independent. Setting $W_a\, \overline{Y}(a) + W_b\, \overline{Y}(b) = (\rho_{ij})$, equation (8) can be written in the form

$$(9) \qquad \sum_{k=1}^{n} \rho_{ik}\, c_k = 0, \quad i = 1, 2, \ldots n.$$

The number of linearly independent solutions of the linear algebraic system (9), (and hence of equation (8)) is, however, precisely equal to the difference between the number of equations, n, and the rank of the determinant $|\rho_{ij}|$. Thus when $|W_a \overline{Y}(a) + W_b \overline{Y}(b)| = 0$, and is of rank $(n - k)$, then there are precisely k linearly independent solutions $C_1\cdot$, $C_2\cdot, \ldots C_k\cdot$, of equation (8), and correspondingly just k linearly independent solutions $Y_i(x)\cdot = \overline{Y}(x)\, C_i\cdot \quad i = 1, 2, \ldots k$, of system (6). Conversely if there are k independent solutions of system (6) the rank of the determinant is $(n - k)$.

Suppose then that system (6) is known to have just k linearly independent solutions. This means that the determinant vanishes and is of rank $(n-k)$, and since $|X_1 U X_2|$ is of the same rank as $|U|$ when $|X_1| \neq 0$, and $|X_2| \neq 0$,[5] it follows that

$$|\overline{Z}(a)\, W_a^{-1} \{ W_a \overline{Y}(a) + W_b \overline{Y}(b) \} \overline{Z}(b)\, W_b^{-1}| = 0$$

and is of rank $(n - k)$, $\overline{Z}(x)$ being any matrix solution of equation (2). If in particular $\overline{Z}(x)$ is chosen as the solution associated with $\overline{Y}(x)$ this reduces to the statement that $|\overline{Z}(a)\, W_a^{-1} + \overline{Z}(b)\, W_b^{-1}| = 0$, and is of rank $(n - k)$, which implies that system (7) also has just k linearly independent solutions. Thus the theorem is proved.

A system of the type (6) or (7) is said to be either *compatible* or *incompatible* according as it does or does not admit of a solution not identically O. Its compatibility is said to be k-fold when the number of its linearly independent solutions is k.

Consider the non-homogeneous system

$$(10) \qquad \begin{cases} Y'(x)\cdot = A(x)\, Y(x)\cdot + B(x)\cdot \\ W_a\, Y(a)\cdot + W_b\, Y(b)\cdot = O. \end{cases}$$

THEOREM: A necessary and sufficient condition that system (10) has a unique solution is that the corresponding homogeneous system (6) is incompatible.

Proof: It was shown (formula (5)) that the general solution of equation (3) is

$$Y(x) \equiv Y_h(x)\, C + \overline{Y}(x)$$

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5 Cf. Bôcher, loc. cit., pp. 77–79.

where $\overline{Y}(x)$ is any particular solution and $Y_h(x)$ is a matrix solution of the homogeneous equation (1). Hence the general solution of the differential equation in (10) is

$$Y(x)\cdot \equiv Y_h(x)\, C\cdot + \overline{Y}\, x)\cdot$$

The substitution of $Y(x)\cdot$ in the boundary conditions shows it to be also a solution of the system (10) provided only that $C\cdot$ satisfies

$$W_a\, \overline{Y}(a) + W_b\, \overline{Y}(b) + \{\, W_a\, Y_h(a) + W_b\, Y_h(b)\}\, C\cdot = O.$$

But this relation can be solved for $C\cdot$, and uniquely determines $C\cdot$ when and only when $|\, W_a\, Y_h(a) + W_b\, Y_h(b)\,| \neq 0$, that is when system (6) is incompatible. Q. E. D.

Assuming then that system (6) is incompatible it is possible to obtain by the following procedure a solution of the equations (10) which is symmetrical with respect to the ends of the given interval $a \leqq x \leqq b$.

Let $Y_1(x)\cdot$ and $Y_2(x)\cdot$ be any pair of solutions of the differential equation

$$Y'(x)\cdot = A(x)\, Y(x)\cdot + \tfrac{1}{2}\, B(x)\cdot. \tag{11}$$

Then clearly their sum $Y_1(x)\cdot + Y_2(x)\cdot = \overline{Y}(x)\cdot$ is a solution of equation (10). But by formula (4), applied to equation (11), a particular choice of $Y_1(x)\cdot$ and $Y_2(x)\cdot$ is seen to be

$$Y_1(x)\cdot = \tfrac{1}{2}\int_a^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt$$

$$Y_2(x)\cdot = \tfrac{1}{2}\int_b^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt,$$

from which it follows that

$$\overline{Y}(x)\cdot = \tfrac{1}{2}\left\{\int_a^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt + \int_b^x Y_h(x)\, Z_h(t)\, B(t)\cdot dt\right\}$$

is a particular solution of the differential equation (10).

Defining $\overline{G}(x, t)$ by the relations

$$\overline{G}(x, t) = \begin{cases} \frac{1}{2}\, Y_h(x)\, Z_h(t) & \text{when } t < x \\ -\frac{1}{2}\, Y_h(x)\, Z_h(t) & \text{when } t > x, \end{cases}$$

we may write

$$\overline{Y}(x)\cdot = \int_a^b \overline{G}(x, t)\, B(t)\cdot\, dt.$$

In accordance with formula (5), therefore, the general solution of the differential equation (10) is given by the equation

$$Y(x)\cdot = \int_a^b \overline{G}(x, t)\, B\,(t)\cdot\, dt + Y_h(x) C\cdot\, .$$

The substitution of this form in the boundary conditions yields the equation

$$\int_a^b \{W_a \overline{G}(a, t) + W_b \overline{G}(b, t)\}\, B(t)\cdot dt + \{W_a\, Y_h(a) + W_b Y_h(b)\} C\cdot\, = O$$

for the constant vector $C\cdot$. Multiplying by the inverse of the matrix

$$\Delta = \{W_a\, Y_h(a) + W_b\, Y_h(b)\},$$

(Δ^{-1} exists since system (6) is incompatible) we see that

$$C\cdot\, = -\,\Delta^{-1} \int_a^b \{W_a \overline{G}\,(a, t) + W_b \overline{G}\,(b, t)\}\, B(t)\cdot dt.$$

It follows that the general solution of system (10) is, in terms of a matrix $G(x, t)$ which is defined by the formula

(12) $$G(x, t) = \overline{G}(x, t) - Y_h(x)\, \Delta^{-1}\, \{W_a\, \overline{G}(a, t) + W_b\, \overline{G}(b, t)\},$$

given by the expression

(13) $$Y(x) = \int_a^b G(x, t)\, B(t)\cdot\, dt.$$

The matrix of functions $G(x, t)$ is known as the *Green's function* for the homogeneous system (6). By a precisely similar method the Green's function $H(x, t)$ for the related system (7) may be derived, the general solution of the non-homogeneous system

$$(14)\qquad \begin{cases} \cdot Z'(x) = -\cdot Z(x)\, A(x) + \cdot D(x) \\ \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0 \end{cases}$$

being given in terms of $H(x, t)$ by the formula

$$(15)\qquad \cdot Z(x) = \int_a^b \cdot D(x)\, H(x, t)\, dt.$$

THEOREM: If $G\,(x, t)$ and $H(x, t)$ are the Green's functions for systems (6) and (7) respectively, then

$$G(x, t) + H(t, x) \equiv 0, \quad t \neq x.$$

Proof:[6] Let $B\cdot$ and $\cdot D$ be any two vectors of the types indicated and consider the two systems

$$Y'\cdot = AY\cdot + B\cdot, \qquad W_a\, Y(a)\cdot + W_b\, Y(b)\cdot = 0,$$
$$\cdot Z' = -\cdot ZA + \cdot D, \qquad \cdot Z(a)\, W_a^{-1} + \cdot Z(b)\, W_b^{-1} = 0.$$

Multiplying the differential equations respectively by $\cdot Z$ on the left and $Y\cdot$ on the right and adding we obtain

$$\cdot ZY\cdot' + \cdot Z'Y\cdot = \cdot ZB\cdot + \cdot DY\cdot,$$

an equality which upon integration yields

$$(16)\qquad \cdot ZY\cdot \Big|_a^b = \int_a^b \cdot Z(t)\, B(t)\cdot dt + \int_a^b \cdot D(x)\, Y(x)\cdot dx.$$

Now from the boundary conditions we have

$$\cdot Z(b) = -\cdot Z(a)\, W_a^{-1}\, W_b, \quad Y(b)\cdot = -W_b^{-1}\, W_a\, Y(a)\cdot,$$

whence

$$\cdot Z(b)\; Y(b)\cdot = \cdot Z(a)\; Y(a)\cdot,$$

i.e.

$$\cdot ZY\cdot \Big|_a^b = 0.$$

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[6] The proof by direct computation is not difficult though somewhat laborious.

But
$$\cdot Z(t) = \int_a^b \cdot D(x)\, H(t, x)\, dx$$

and
$$Y(x)\cdot = \int_a^b G\,(x, t)\, B(t)\cdot dt$$

from formulas (13) and (15).
Hence equation (16) is equivalent to

$$\int_a^b \int_a^b \cdot D(x)\,\{H\,(t, x) + G(x, t)\}\, B(t)\cdot dx\, dt = O. \tag{17}$$

This result, having been obtained without reference to the nature of $B\cdot$ and $\cdot D$ must, moreover, hold for all possible choices of these vectors. We shall proceed to choose a particular set apposite to the proof in hand.

Let any point (x_0, t_0) of the region $a \leqq x, t \leqq b, x \neq t$, be arbitrarily chosen and draw the surrounding small rectangle Δs whose sides are $t = t_0 \pm \Delta\, t$ and $x = x_0 \pm \Delta\, x$. Then choose $B\cdot$ and $\cdot D$ so that
$b_l\,(t) \equiv 0$, when $l \neq j_0$, $\quad b_{j_0}(t) \neq 0$ in Δs, $\quad b_{j_0}(t) \equiv 0$ outside of Δs,
$d_k(x) \equiv 0$, when $k \neq i_0$, $\quad d_{i_0}(x) \neq 0$ in Δs, $\quad d_{i_0}(x) \equiv 0$ outside of Δs.

For this choice equation (17) is equivalent to

$$\int_{t_0-\Delta t}^{t_0+\Delta t} \int_{x_0-\Delta x}^{x_0+\Delta x} d_{i_0}(x)\,\{h_{i_0j_0}(t, x) + g_{i_0j_0}(x, t)\}\, b_{j_0}(t)\, dxdt = 0, \tag{18}$$

and inasmuch as $d_{i_0}(x)\, b_{j_0}(t) \neq 0$ in Δs, clearly $\{h_{i_0j_0}(t, x) + g_{i_0j_0}(x, t)\}$ changes sign, i.e. vanishes somewhere in this region. Now let Δx and Δt approach zero. Then in the limit we have

$$h_{i_0j_0}(t_0, x_0) + g_{i_0j_0}(x_0, t_0) = 0.$$

From this it is seen that $G(x_0, t_0) + H(t_0, x_0) = O$, and since (x_0, t_0) was any point not on the diagonal it follows that

$$G(x, t) + H(t, x) \equiv O, \quad x \neq t. \qquad \text{Q. E. D.}$$

By direct reference to formula (12) it is readily seen that the Green's function possesses the following characteristics:

i) The elements of $G(x, t)$ are continuous in x except for $x = t$. Along this line $(x = t)$ there is a jump of unit magnitude in the elements of the principal diagonal, i.e.

$$G(x, x - 0) - G(x, x + 0) = I.$$

ii) For any given t, $G(x, t)$ satisfies equation (6) in x, except along the line $x = t$, i.e.

$$\frac{\partial G(x, t)}{\partial x} \equiv A(x)\, G(x, t), \quad x \neq t.$$

iii) For any given t, $G(x, t)$ satisfies the boundary conditions of system (6) in x, i.e.

$$W_a G(a, t) + W_b\, G(b, t) \equiv O.$$

Conversely we have the

THEOREM: The dependence of $G(x, t)$ upon the variables x, t is completely determined by the characteristics (i) to (iii) above.

Proof: Suppose $\overline{G}\,(x, t)$ is a matrix possessing the characteristics (i), (ii), and (iii). By (i), (ii), $J(x, t)$ defined by $J(x, t) \equiv \overline{G}\,(x, t) - G(x, t)$ is continuous for all x, $a \leqq x \leqq b$. Moreover, $J(x, t)$ satisfies (iii) and is therefore a solution of system (6) in x. But system (6) is incompatible by hypothesis. Hence $J(x, t) \equiv O$

and $$\overline{G}(x, t) \equiv G(x, t). \qquad \text{Q. E. D.}$$

It is readily verified that a further set of three characteristics which completely determines the dependence $G(x, t)$ upon the variables may be obtained by interchanging x and t and replacing system (6) by system (7) in the discussion above.

The fact established by this theorem should be carefully noted. While the choice of the pair of associated solutions Y_h and Z_h on the right-hand side of equation (12) is not unique, yet the entire function $G(x, t)$ is independent of that choice.

Section V.

The formal solutions of the equation

$Y'(x)\cdot = \{A(x)\lambda + B(x)\} Y(x)\cdot$.

Returning to the homogeneous equation let us consider the nature of the solutions when the matrix coefficient of $Y\cdot$ is made to depend linearly upon a parameter λ which is free to take on all values in the finite complex λ plane. To this end we shall study the equation

$$(19) \qquad Y'(x)\cdot = \{ A(x)\lambda + B(x) \} Y(x)\cdot,$$

where $A(x)$ and $B(x)$ are matrices of continuous functions, by making the assumption that the equation has a formal solution

$$(20) \quad Y(x)\cdot \equiv e^{\lambda \int_\alpha^x \gamma(x)dx} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \frac{1}{\lambda^2} P_2(x)\cdot + \dots \right\}$$

where α is any chosen constant.

If $n = 1$, equation (19) can be directly integrated and is seen to have an actual solution of the form (20). The passage to formulas (45) can, therefore, be made directly in this case, and hence we shall assume in the intervening work that $n \geqq 2$.

Setting $\int_\alpha^x \gamma(x)\, dx = \Gamma(x)$, and substituting the form (20) in the equation (19), we obtain the formal identity

$$\lambda\gamma(x)e^{\lambda\Gamma(x)} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \dots \right\} + e^{\lambda\Gamma(x)} \left\{ P_0'(x)\cdot + \dots \right\}$$
$$\equiv \{A(x)\lambda + B(x)\}\, e^{\lambda\Gamma(x)} \left\{ P_0(x)\cdot + \frac{1}{\lambda} P_1(x)\cdot + \dots \right\},$$

from which it is seen, upon equating the coefficients of λ, that

$$\gamma(x)\, P_0(x)\cdot \equiv A(x)\, P_0(x)\cdot.$$

This is satisfied by a vector $P_0(x)\cdot$ not all of whose elements vanish, when and only when

$$(21) \qquad | a_{ij}(x) - \delta_{ij}\,\gamma(x) | = 0,$$

i.e. when $\gamma(x)$ satisfies equation (21). For any given x, say $x = x_0$, however, the left-hand side of (21) is a polynomial of degree n in $\gamma(x_0)$. Hence the equation is satisfied by n roots $\gamma_1(x_0), \gamma_2(x_0), \ldots \gamma_n(x_0)$. We shall assume that for the case in hand these roots can be grouped into the n functions $\gamma_1(x), \gamma_2(x), \ldots \gamma_n(x)$, which satisfy the three conditions

$$(22) \quad \begin{cases} \text{(i)} & \gamma_i(x) \text{ continuous}, \quad i = 1, 2, \ldots n, \quad a \leqq x \leqq b, \\ \text{(ii)} & \gamma_j(x) \neq \gamma_i(x) \text{ for } j \neq i, \\ \text{(iii)} & \gamma_i(x) \neq 0, \quad i = 1, 2, \ldots n. \end{cases}$$

Clearly the last condition can be fulfilled only if $| A | \neq 0$; we shall assume this to be the case.

Consider now a change of the dependent variable in equation (19). Setting $Y(x) \cdot = \Phi(x) \overline{Y}(x) \cdot$, where $\Phi(x)$ is a matrix whose elements are continuous as well as their first derivatives, $a \leqq x \leqq b$, and $| \Phi | \neq 0$, the equation becomes $\Phi' \overline{Y} \cdot + \Phi \overline{Y} \cdot' = \{A\lambda + B\} \Phi Y \cdot$, that is $\overline{Y}' \cdot = \{\Phi^{-1} A \Phi \lambda + \Phi^{-1} B \Phi - \Phi^{-1} \Phi'\} \overline{Y} \cdot$ It is evident here that if the coefficient of λ is not originally I then no such change of variable has the effect of reducing it to I. In the subsequent work it is desirable to transform $A(x)$ into the matrix $R(x)$ given by $R(x) = (\delta_{ij} \gamma_j(x))$. By such a transformation the formal solutions (20) are carried into others with the same functions $\gamma_i(x)$ in the exponents.

Since the $\Phi(x)$ in question must satisfy $\Phi^{-1} A \Phi \equiv R$,[7] it must fulfill the conditions

$$\text{(i)} \qquad A\Phi \equiv \Phi R,$$

$$\text{(ii)} \qquad | \Phi(x) | \neq 0.$$

To satisfy condition (i) the elements of Φ must be solutions of the algebraic system

$$\sum_{k=1}^{n} a_{ik} \varphi_{kj} = \varphi_{ij} \gamma_j, \quad i, j, = 1, 2, \ldots n,$$

while the possibility of fulfilling this condition by means of a matrix $\Phi(x)$, none of whose columns contain only vanishing elements, depends upon the existence of a solution for each of the n linear systems

[7] For fuller discussion see, for instance, Bôcher, loc. cit. Chap. XXI.

$$\sum_{k=1}^{n} \{a_{ik} - \delta_{ik}\, \gamma_{j_0}\}\, \varphi_{kj_0} = 0, \tag{23}$$

obtained by giving to j_0 successively the values $1, 2, \ldots n$. This means that each of the determinants $| a_{ik} - \delta_{ik}\, \gamma_{j_0} |$, $j_0 = 1, 2, \ldots n$ must vanish, and inasmuch as the functions $\gamma_j(x)$ were chosen as the roots of equation (22), which is precisely the condition that the determinants in question do vanish, the existence of a matrix $\Phi(x)$ of the type desired is established.

Condition (ii) requires further, however, that this matrix Φ be one for which $| \Phi(x) | \neq 0$.

Suppose $| \Phi(x) | = 0$. Then the system of linear equations $\sum_{k=1}^{n} \varphi_{ik}\, v_k = 0$ admits of solution by means of a set of quantities v_1, $v_2, \ldots v_n$, at least one of which differs from zero,[8] i.e. there exists a vector $V \cdot \not\equiv O$, such that $\Phi V \cdot \equiv O$. But then $A\Phi V \cdot \equiv O$, and in view of the relation (i) it follows further that $\Phi R V \cdot \equiv O$. By continued repetition of this reasoning it can be shown likewise that $\Phi R^k V \cdot \equiv O$ for all values of k. Consequently $\Phi\{c_0 I + c_1 R + c_2 R^2 + \ldots + c_{n-1} R^{n-1}\} V \cdot \equiv O$. But on the hypothesis that $\gamma_j \neq \gamma_i$ when $j \neq i$ it is well known that the determinant

$$\begin{vmatrix} 1 & \gamma_1 & \gamma_1^2 & \ldots & \gamma_1^{n-1} \\ 1 & \gamma_2 & \gamma_2^2 & \ldots & \gamma_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \gamma_n & \gamma_n^2 & \ldots & \gamma_n^{n-1} \end{vmatrix} \neq 0.$$

Hence the system of equations

$$\begin{array}{c} c_0 + c_1\gamma_1 + c_2\gamma_1^2 + \ldots + c_{n-1}\, \gamma_1^{n-1} = k_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_0 + c_1\gamma_n + c_2\gamma_n^2 + \ldots + c_{n-1}\, \gamma_1^{n-1} = k_n, \end{array}$$

admits of a solution in the c's not all zero for any choice of the quantities k_i not all zero. Some choice of the set $c_0, c_1 \ldots c_{n-1}$, can therefore always be made to satisfy the relation

$$\{c_0 I + c_1 R + \ldots + c_{n-1} R^{n-1}\} \equiv (\delta_{ij}\, k_j).$$

[8] Cf. Bôcher, loc. cit., pp. 47.

Then $$\Phi(\delta_{ij}\, k_j)\, V\cdot \equiv 0,$$

i.e. $\sum_{l=1}^{n} \varphi_{il}\, k_l\, v_l \equiv 0$; for all choices of the set $k_1, k_2, \ldots k_n$. But by construction some v, say v_{j_0} is not zero. If then the k's are chosen so that $k_i = 0,\ i \neq j_0,\ k_{j_0} \neq 0$, it follows that $\varphi_{ij_0} = 0$ for all i. Inasmuch as no column of Φ contains only vanishing elements this involves a contradiction. Hence the hypothesis $|\Phi(x)| = 0$ is not tenable, and the Φ in question fulfills condition (ii), i.e. $|\Phi(x)| \neq 0$.

By a change of variable, then, equation (19) can be given the form $\overline{Y}'\cdot = \{R\lambda + \overline{B}\}\, \overline{Y}\cdot$, where the matrix $\overline{B}$, being given by the relation $\overline{B} = \Phi^{-1} B\Phi - \Phi^{-1}\Phi'$, is a matrix of continuous functions. Supposing this to have been done we may drop the dashes and consider the equation in the form

$$Y'(x)\cdot = \{R(x)\lambda + B(x)\}\, Y(x)\cdot. \tag{24}$$

Since we have found that the $Y(x)\cdot$ of (20) may be any one of the n vectors $Y_i(x)\cdot$ obtained by replacing $\Gamma(x)$ by $\Gamma_i(x)$ the further discussion might well be carried through for each of these vectors individually. However, if the matrix $E(x)$ is defined by the relation

$$E(x) \equiv (\delta_{ij}\, e^{\lambda\Gamma_j(x)}),$$

it is readily seen that the matrices $P_i(x)$ can be chosen so that the j^{th} column of the matrix

$$Y(x) \equiv \left\{ P_0(x) + \frac{1}{\lambda}\, P_1(x) + \ldots \right\} E(x) \tag{25}$$

is precisely the general column of the vector $Y_j(x)\cdot$ Hence all cases are simultaneously treated by the consideration of those formal solutions of the matrix equation

$$Y'(x) = \{R(x)\lambda + B(x)\}\, Y(x) \tag{26}$$

which have the form (25).

Inasmuch as $E'(x) = (\lambda\gamma_j(x)\, \delta_{ij}\, e^{\lambda\Gamma_j(x)}) = \lambda\, R(x)\, E(x)$, the formal substitution of (25) in (26) gives the identity

$$(27)\qquad \left\{P_0+\frac{1}{\lambda}P_1+\frac{1}{\lambda^2}P_2+\ldots\right\}\lambda RE+\left\{P_0'+\frac{1}{\lambda}P_1'+\ldots\right\}E\equiv$$
$$\left\{\lambda R+B\right\}\left\{P_0+\frac{1}{\lambda}P_1+\ldots\right\}E.$$

Equating the coefficients of λ we obtain the relation

$$P_0R = RP_0,$$

i.e.

$$p_{ij}^{(0)}\,\gamma_j = \gamma_i\,p_{ij}^{(0)},$$

whence it is seen that $p_{ij}^{(0)} = 0$, when $j \neq i$. Equating the coefficients of λ^0 we obtain in similar manner the equation

$$P_1R + P_0' = RP_1 + BP_0,$$

i.e.

$$(28)\qquad p_{ij}^{(1)}\,\gamma_j + p_{ij}^{(0)\prime} = \gamma_i\,p_{ij}^{(1)} + \sum_{k=1}^{n} b_{ik}\,p_{kj}^{(0)},$$

whence it is seen, upon setting $i = j$ that

$$p_{jj}^{(0)\prime} = \sum_{k=1}^{n} b_{jk}\,p_{kj}^{(0)} = b_{jj}\,p_{jj}^{(0)},$$

namely, that $p_{jj}^{(0)} = f_j\,e^{\int_\alpha^x b_{jj}dx}$.

Having obtained in this manner all the elements of the matrix P_0 we find further from the equation (28) for $i \neq j$ that

$$(29)\qquad p_{ij}^{(1)} = \frac{b_{ij}\,p_{jj}^{(0)}}{\gamma_j-\gamma_i}.$$

Moreover, equating the coefficients of $\frac{1}{\lambda}$ we see that

$$P_2R + P_1' = RP_2 + BP_1,$$

i.e.

$$p_{ij}^{(2)}\,\{\gamma_j-\gamma_i\} + p_{ij}^{(1)\prime} = \sum_{k=1}^{n} b_{ik}\,p_{kj}^{(1)},$$

whence it follows upon setting $i = j$ that

$$(30)\qquad p_{jj}^{(1)\prime} = \sum_{k=1}^{n} b_{jk}\,p_{kj}^{(1)}.$$

Since the other quantities under the sign of summation in this formula have all been determined, $p_{jj}^{(1)}$ may be found by means of a quadrature. When this has been done the matrix P_1 has been completely determined.

It should be observed that we have thus far required only that the matrices $R(x)$ and $B(x)$ be continuous. To determine the elements of P_2 however, we have first, for $i \neq j$, the equation

$$p_{ij}^{(2)} = \frac{-p_{ij}^{(1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(1)}}{\gamma_j - \gamma_i},$$

and since this formula implies the existence of $p_{ij}^{(1)\prime}$, whereas it is seen from (29) that $p_{ij}^{(1)}$ has a derivative only if this is true of the matrices $R(x)$ and $B(x)$, we cannot proceed to the determination of P_2 if $R(x)$ and $B(x)$ are merely continuous.

Let us suppose then that $R(x)$ and $B(x)$ both possess continuous derivatives up to and including those of order $k \geqq 1$, but that perhaps one of these matrices possesses no such derivative of order $(k+1)$. If, in particular $R(x)$ and $B(x)$ possess infinitely many derivatives we may take $k = \infty$. The derivatives of P_1 up to and including that of order k are now seen to exist from formulas (29) and (30).

Equating formally the coefficients of $\dfrac{1}{\lambda^{\mu-1}}$ in the identity (27) we have

$$P_\mu R + P'_{\mu-1} = RP_\mu + BP_{\mu-1},$$

i.e.

$$p_{ij}^{(\mu)}\{\gamma_j - \gamma_i\} = -p_{ij}^{(\mu-1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(\mu-1)},$$

whence
$$p_{ij}^{(\mu)} = \frac{-p_{ij}^{(\mu-1)\prime} + \sum_{k=1}^{n} b_{ik}\, p_{kj}^{(\mu-1)}}{\gamma_j - \gamma_i} \quad \text{for } i \neq j,$$

and again, equating the coefficients of $\dfrac{1}{\lambda^{\mu}}$ we have

$$P_{\mu+1} R + P'_\mu = RP_{\mu+1} + BP_\mu$$

i.e. $$p_{ij}^{(\mu+1)}\{\gamma_j-\gamma_i\}+p_{ij}^{(\mu)\prime}=\sum_{k=1}^{n} b_{ik}\,p_{kj}^{(\mu)},$$

whence it is seen, upon setting $i=j$ that

$$p_{jj}^{(\mu)\prime}=\sum_{k=1}^{n} b_{jk}p_{kj}^{(\mu)},$$

namely that $p_{jj}^{(\mu)}$ can be determined by means of a quadrature if the quantities on the right are known.

The determination of the elements of $p_{jj}^{(\mu)}$ by means of these formulas depends, therefore, only upon a knowledge of the elements of $P_{\mu-1}$ and upon the existence of $P'_{\mu-1}$. Moreover, it is seen that in general P_μ possesses one less derivative than $P_{\mu-1}$. Inasmuch as P_1 has already been determined, and was seen to possess k derivatives it is clear that the matrices P_μ for $\mu = 0, 1, \ldots, (k+1)$, may be successively determined, and that in general the matrix P_{k+1} is merely continuous.

If k is finite, we can, therefore, determine a differentiable matrix $S(x)$, given by

$$S(x)\equiv\left\{P_0(x)+\frac{1}{\lambda}P_1(x)+\ldots+\frac{1}{\lambda^k}P_k(x)\right\}E(x) \tag{31}$$

$$\equiv\left(e^{\lambda\Gamma_j(x)}\left\{p_{ij}^{(0)}+\frac{p_{ij}^{(1)}}{\lambda}+\ldots+\frac{p_{ij}^{(k)}}{\lambda^k}\right\}\right),$$

which will satisfy the equation

$$S'(x)=\{\lambda R+B\}\,S(x)+\frac{1}{\lambda^k}\{P_k'(x)-B(x)\,P_k(x)\}\,E(x),$$

i.e. an equation of the form

$$S'(x)=\left\{\lambda R(x)+B(x)+\frac{1}{\lambda^k}L(x,\lambda)\right\}S(x), \tag{32}$$

where $L(x, \lambda)$ is a matrix each element of which is rational in λ, with coefficients continuous in x, given by power series in $\left(\frac{1}{\lambda}\right)$.

If on the other hand $k=\infty$, as many terms of the infinite series

$$p_{ij}=p_{ij}^{(0)}+\frac{p_{ij}^{(1)}}{\lambda}+\frac{p_{ij}^{(2)}}{\lambda^2}+\ldots.$$

as are desired may be used. These series are not in general convergent. Nevertheless the formal matrix

$$S(x) \equiv \left\{ P_0(x) + \frac{1}{\lambda} P_1(x) + \ldots\ldots \right\} E(x)$$
$$\equiv \left(e^{\lambda \Gamma_j(x)} \left\{ p_{ij}^{(0)} + \frac{p_{ij}^{(1)}}{\lambda} + \ldots \right\} \right)$$

which is found in this case, formally satisfies the equation (26).

With the convention that the term in $\dfrac{1}{\lambda^k}$ is to be omitted if $k = \infty$ formula (32) holds in all cases. $S(x)$ will be called a formal matrix solution of equation (26) regardless of whether k is, for the case in hand, finite or infinite.

It should be observed that each $p_{ij}^{(\mu)}$ is not wholly determined but contains a single arbitrary constant of integration, independent of x. This arbitrariness corresponds to the fact that any convergent power series c_j in $\left(\dfrac{1}{\lambda}\right)$ with constant coefficients may be multiplied into each column of $S(x)$ without thereby destroying its property of being a formal solution in the sense (32).

Using the notation $[\psi_0]_k$, or $[\psi_0]$, for an expression of the form

$$\psi_0 + \frac{\psi_1}{\lambda} + \ldots\ldots + \frac{\psi_k}{\lambda^k} + \frac{\psi(x,\lambda)}{\lambda^{k+1}}$$

where ψ is bounded for $|\lambda|$ large, we have

$$S(x) = \left(\left[\delta_{ij}\, c_j e^{\int_\alpha^x b_{jj} dx} \right]_k \right) E(x).$$

It is clear that an alternative form is

$$(33) \qquad S(x) = \left(c_j\, e^{\lambda \Gamma_j(x) + B_j(x)}\, [\delta_{ij}]_k \right),$$

where $B_j(x) = \displaystyle\int_\alpha^x b_{jj}\, dx$.

By similar considerations the equation $Z' = -Z\{A\lambda + B\}$ may be transformed and a formal matrix solution $T(x)$ for the resulting equation

$$Z'(x) = -Z(x)\{R(x)\lambda + B(x)\} \tag{34}$$

be obtained. This $T(x)$ has the form

$$T(x) = \left(\bar{c}_i\, e^{-\lambda\Gamma_i(x)-B_i(x)}\,[\delta_{ij}]_k\right), \tag{35}$$

and satisfies an equation of the form

$$T'(x) = -T(x)\left\{\lambda R(x) + B(x) + \frac{1}{\lambda^k}M(x, \lambda)\right\}.$$

Moreover each row is a formal solution of the vector equation

$$\cdot Z'(x) = -\cdot Z(x)\{R(x)\,\lambda + B(x)\}. \tag{36}$$

Considering differentiation as merely a formal process defined by the usual rules it is readily seen that the differentiation of the formal matrices $S(x)$ and $T(x)$ is permissible. Hence we have

$$\begin{aligned} \frac{d}{dx}TS &= TS' + T'S \\ &= T\left\{\lambda R + B + \frac{1}{\lambda^k}L\right\}S - T\left\{\lambda R + B + \frac{1}{\lambda^k}M\right\}S \\ &= \frac{1}{\lambda^k}T\{L - M\}S. \end{aligned}$$

Since

$$\begin{aligned} TS &\equiv \left(\sum_{h=1}^{n}\bar{c}_i\, e^{-\lambda\Gamma_i(x)-B_i(x)}\,[\delta_{ih}]_k\, c_j\, e^{\lambda\Gamma_j(x)+B_j(x)}\,[\delta_{hj}]_k\right) \\ &\equiv \left(\bar{c}_i\, c_j\, e^{\lambda\{\Gamma_j(x)-\Gamma_i(x)\}+B_j(x)-B_i(x)}\left\{\delta_{ij} + \frac{\sigma_{ij}^{(1)}}{\lambda} + \ldots + \frac{\sigma^{(2k)}}{\lambda^{2k}}\right\}\right), \end{aligned}$$

we find upon differentiating, and removing the exponential factor,

$$\begin{aligned} \bar{c}_i\, c_j\Bigg\{\{\lambda(\gamma_j-\gamma_i) + b_{jj}-b_{ii}\}\left\{\delta_{ij} + \frac{\sigma_{ij}^{(1)}}{\lambda} + \ldots + \frac{\sigma_{ij}^{(2k)}}{\lambda^{2k}}\right\} \\ + \left\{\frac{\sigma_{ij}^{(1)\prime}}{\lambda} + \ldots + \frac{\sigma_{ij}^{(2k)\prime}}{\lambda^{2k}}\right\}\Bigg\} \equiv \end{aligned}$$

$$\frac{1}{\lambda^k} \sum_{h,s=1}^{n} [\delta_{ih}]_k \{l_{hs} - m_{hs}\} [\delta_{sj}]_k.$$

Equating the coefficients of λ^0 we have

$$\bar{c}_i\, c_j \{ (\gamma_j - \gamma_i)\, \sigma_{ij}^{(1)} + (b_{jj} - b_{ii})\, \delta_{ij}\} \equiv 0,$$

from which it follows that $\sigma_{ij}^{(1)} = 0$ when $i \neq j$. Again, equating the coefficients of $\frac{1}{\lambda}$ we have

$$(\gamma_j - \gamma_i)\, \sigma_{ij}^{(2)} + (b_{jj} - b_{ii})\, \sigma_{ij}^{(1)} + \sigma_{ij}^{(1)\prime} \equiv 0,$$

from which it is seen, upon setting $i = j$ that $\sigma_{jj}^{(1)\prime} \equiv 0$, namely that $\sigma_{jj}^{(1)} \equiv$ constant. The relation shows on the other hand that $\sigma_{ij}^{(2)} = 0$ when $i \neq j$. Equating to zero successively the coefficient of each individual power of $\frac{1}{\lambda}$ it is found in the same way that $\sigma_{ij}^{(\mu)} \equiv 0$ when $i \neq j$, $\sigma_{jj}^{(\mu)} \equiv$ constant for $\mu = 1, 2, \ldots, (k-1)$. It follows that

$$TS = \left(e^{\lambda\{\Gamma_j - \Gamma_i\} + B_j - B_i} \left\{ \bar{c}_i\, c_j\, \delta_{ij} \left\{ 1 + \frac{\sigma_{jj}^{(1)}}{\lambda} + \ldots + \frac{\sigma_{jj}^{(k-1)}}{\lambda^{k-1}} \right\} + \frac{\sigma_{ij}(x, \lambda)}{\lambda^k} \right\} \right)$$

$$= (\bar{c}_j\, c_j\, \delta_{ij}\, [1]_{k-1}) + \frac{1}{\lambda^k} E^{-1} (\sigma_{ij})\, E,$$

where the coefficients of $[1]_{k-1}$ are constants.

Now for any choice of the set of series c_j it is clearly possible to choose a set $\bar{c}_j$ such that $\bar{c}_j\, c_j\, [1]_{k-1} = 1$, $j = 1, 2, \ldots n$. The formal solutions $S(x)$ and $T(x)$ corresponding respectively to these values of c_j and $\bar{c}_j$ are closely related. They are called *associated formal* solutions and satisfy the relation $T(x)\, S(x) = I + \frac{1}{\lambda^k} E^{-1} (\sigma_{ij})\, E$.

Since in particular c_j may be chosen as $c_j = 1$ it is seen that there exists a formal solution having the form

$$S(x) = (e^{\lambda \Gamma j(x) + B j(x)} [\delta_{ij}]). \tag{37}$$

In accordance with the definition above the associated solution is given by the set of $\bar{c}_j$'s which satisfy the relations $\bar{c}_j = 1$. It is

apparent, therefore, that the solution associated with (37) is of the form

$$T(x) = (e^{-\lambda\Gamma_i(x)-B_{ij}(x)}\,[\delta_{ij}]). \tag{38}$$

SECTION VI.

The relation of the formal solutions to the actual solutions.

It was observed in the preceding section that the formal matrix $S(x)$ either satisfies equation (26) only in an approximate sense (i.e. satisfies (32)) or, if $k = \infty$ satisfies it only formally, since the elements of $S(x)$ are in that case infinite series which are not in general convergent. $S(x)$ is, therefore, not a matrix solution of equation (26), and its significance requires further investigation.

Consider the actual matrix which is derived from $S(x)$ by retaining in the latter only the first $(m + 1)$ terms of its elements, where m is any positive integer not exceeding k. This matrix $\overline{S}(x)$ may be written, in accordance with formula (37),

$$\begin{aligned}\overline{S}(x) &= (e^{\lambda\Gamma_j(x)+B_j(x)}\,[\delta_{ij}]_m)\\ &= \overline{P}(x, \lambda)\, E(x),\end{aligned}$$

where
$$\overline{P}(x,\lambda) = (e^{B_j(x)}\,[\delta_{ij}]_m),$$

and is seen to be analytic in λ. Since $|\, e^{B_j(x)}\delta_{ij} \,| \neq 0$, it follows that $|\, \overline{S} \,| \neq 0$ for $|\, \lambda \,| > N$. If the formal solution $S(x)$ is written as $P(x, \lambda)\, E(x)$, it is apparent that we have the formulas

$$\overline{S}(x) = \left\{ I - \frac{1}{\lambda^{m+1}}\,[Q]\,P^{-1} \right\} S(x),$$

$$S(x) = \left\{ I + \frac{1}{\lambda^{m+1}}\,[Q]\,\overline{P}^{-1} \right\} \overline{S}(x),$$

from which we obtain upon differentiating and substituting from the equation (32), the relation

$$\overline{S}'(x) = \left\{ \left\{ I - \frac{1}{\lambda^{m+1}} [Q] P^{-1} \right\} \left\{ \lambda R + B + \frac{1}{\lambda^k} L \right\} - \frac{1}{\lambda^{m+1}} ([Q] \overline{P}^{-1})' \right\}$$
$$\left\{ I + \frac{1}{\lambda^{m+1}} [Q] \overline{P}^{-1} \right\} \overline{S}(x).$$

It is apparent from this that $\overline{S}(x)$ is a solution of a homogeneous differential matrix equation of the type

$$\overline{S}'(x) = \left\{ \lambda R + B + \frac{1}{\lambda^m} \Phi(x, \lambda) \right\} \overline{S}(x). \tag{39}$$

The elements of $\Phi(x, \lambda)$ are power series in $\left(\frac{1}{\lambda}\right)$ with coefficients which are continuous in x, and are seen to be convergent, since these elements are rational in λ.

But equation (26) can be written in the form

$$Y'(x) = \left\{ R(x)\lambda + B(x) + \frac{1}{\lambda^m} \Phi(x, \lambda) \right\} Y(x) - \frac{1}{\lambda^m} \Phi(x, \lambda)\, Y(x), \tag{40}$$

and considering this as a non-homogeneous equation we know, in virtue of the developments of page 62, that its solutions, i.e. the solutions of (26), are given by

$$Y(x) = \overline{S}(x)\, C + \int^x \overline{S}(x)\, \overline{T}(t) \left\{ -\frac{1}{\lambda^m} \Phi(t, \lambda)\, Y(t) \right\} dt,$$

i.e. by

$$Y(x) = \overline{S}(x)\, C - \frac{1}{\lambda^m} \int^x \overline{S}(x)\, \overline{T}(t)\, \Phi(t, \lambda)\, Y(t)\, dt, \tag{41}$$

where $\overline{T}(x) \equiv \overline{S}^{-1}(x)$ and where the lower limit of integration, which has been omitted, may be chosen at pleasure for each of as many parts of the integrand as desired.

Substituting for $Y(x)$ in equation (41) its equivalent as given by the form

$$Y(x) = U(x)\, \overline{S}(x), \tag{42}$$

we have further

$$(43)\quad U(x) = \bar{S}(x)\, C\bar{T}(x) - \frac{1}{\lambda^m}\int^x \bar{S}(x)\, \bar{T}(t)\Phi(t, \lambda)\, U(t)\bar{S}(t)\, \bar{T}(x)\, dt.$$

THEOREM: If the functions $\gamma_i(x)$, $i = 1, 2, \ldots n$, satisfy the relations $\arg\{\gamma_j(x) - \gamma_i(x)\} = h_{ij}$, $i, j, = 1, 2, \ldots n$, where each h_{ij} is a constant, then there corresponds to each sector bounded by two adjacent rays $R\{\lambda\{\gamma_j(x) - \gamma_i(x)\}\} = 0$ [8] a choice of the lower limits of integration which is such that for λ within the sector and $|\lambda| > N$, and for any continuous matrix $U(t)$, each element of the matrix

$$\psi(x, \lambda) \equiv \int^x \bar{S}(x)\, \bar{T}(t)\, \Phi(t, \lambda)\, U(t)\, \bar{S}(t)\, \bar{T}(x)\, dt$$

is less numerically than KM, where M is the largest numerical maximum attained by any element of U.

Proof: Writing

$$\bar{S}(x) = (s_{ij}(x))\, E(x), \quad \bar{T}(x) = E^{-1}(x)\, (t_{ij}(x)),$$

we have

$$\psi(x, \lambda) = \left(\sum_{h, l, p, q, r=1}^{n} \int^x s_{ih}(x) e^{\lambda\{\Gamma_h(x) - \Gamma_h(t)\}} t_{hl}(t)\, \varphi_{lp}(t, \lambda)\, u_{pq}(t) \right.$$

$$\left. s_{qr}(t)\, e^{\lambda\{\Gamma_r(t) - \Gamma_r(x)\}}\, t_{rj}(x)\, dt \right)$$

$$= \sum_{h, r}^{n} \left(\int^x e^{\lambda \int_t^x \{\gamma_h(\xi) - \gamma_r(\xi)\} d\xi}\, \omega_{ij}^{(hr)}(t, x, \lambda)\, dt \right),$$

where, for $|\lambda| > N$, $|\omega_{ij}^{(hr)}| < \kappa^{(hr)} M$ for all i and j, $k^{(hr)}$ being a positive constant.

Consider a particular sector and any element

$$\int^x e^{\lambda \int_t^x \{\gamma_h(\xi) - \gamma_r(\xi)\} d\xi}\, \omega_{ij}^{(hr)}\, dt.$$

If $R\{\lambda\{\gamma_h(\xi) - \gamma_r(\xi)\}\} \leqq 0$ for λ within the sector and any ξ it is so for all ξ, and, provided that $t \leqq x$, the integral

[8] The notation $R\{\varphi\}$ is used to indicate "the real part of φ."

$$\int_a^x e^{\lambda \int_t^x \{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi} \omega_{ij}^{(hr)}\, dt$$

is less numerically than $k^{(hr)}M$. However, if $R\{\lambda\{\gamma_h(\xi) - \gamma_r(\xi)\}\} > 0$ for λ within the sector and any ξ, it is so for all ξ, and, provided that $t \geqq x$, the integral

$$\int_b^x e^{\lambda \int_t^x \{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi} \omega_{ij}^{(hr)}\, dt$$

is similarly bounded. Consequently if $\kappa_{hr}(\lambda)$ is defined by the relations

$$\kappa_{hr} = a \text{ if } R\{\lambda\{\gamma_h(\xi) - \gamma_r(\xi)\}\} \leqq 0,\ \kappa_{hr} = b \text{ otherwise,}$$

the numerical value of each element of the matrix

$$\psi(x, \lambda) = \left(\sum_{h,r=1}^{n} \int_{\kappa_{hr}}^{x} e^{\lambda \int_t^x \{\gamma_h(\xi)-\gamma_r(\xi)\}d\xi} \omega_{ij}^{(hr)}\, dt \right)$$

is clearly less than KM, where $K = (b - a) \sum_{h,r=1}^{n} k^{(hr)}$, for λ within the sector and $|\lambda| > N$. Q. E. D.

Assuming the limits chosen in the manner above, a $\psi(x, \lambda)$ corresponding to each sector and to each U is uniquely determined. Moreover, we have

$$U(x) = \overline{S}(x)\, C\overline{T}(x) - \frac{1}{\lambda^m}\psi(x, \lambda). \tag{44}$$

Consider now the particular solution, $Y_0(x)$, of equation (26) which satisfies the relation $Y_0(a) = \overline{S}(a)$. Since $\overline{S}(x)$ as well as the coefficients of equation (26) are analytic in λ, $Y_0(x)$ is likewise analytic in λ. Moreover we know from page 56 that every solution of the equation is of the form $Y(x) = Y_0(x)\, D$, where the elements of D are constants with respect to x. Substituting this form of $Y(x)$ into equation (41) and fixing x and solving for C we obtain the relation

$$C = \overline{T}(x_0) \left\{ Y_0(x_0) + \frac{1}{\lambda^m} \int^{x_0} \overline{S}(x_0)\, \overline{T}(t)\, \Phi(t, \lambda)\, Y_0(t)\, dt \right\} D,$$

i.e.
$$c_{ij} = \sum_{h=1}^{n} \rho_{ih} \, d_{hj} \, ,$$

where the quantities ρ_{ih} are analytic in λ.

Inasmuch as $Y(x)$ is a non-identically zero solution if $D \neq O$ there will exist such a solution for which $C = O$ provided the determinant $|\rho_{ih}|$ vanishes. If on the other hand this determinant does not vanish then there exists a solution $Y(x) \not\equiv O$ corresponding to every $C \neq O$.

Suppose the determinant $|\rho_{ih}| = 0$. Then there corresponds to the choice $C = O$ a solution $\overline{Y}(x)$ for which the matrix $\overline{U}(x)$ defined by $\overline{Y}(x) = \overline{U}(x)\, \overline{S}(x)$ satisfies the relation

$$\overline{U}(x) = -\frac{1}{\lambda^m}\left(\sum_{h,r=1}^{n} \int_{\kappa_{h,r}}^{x} e^{\lambda \int_t^x \{\gamma_h(\xi)-\gamma_r(\xi)\}\,d\xi} \omega_{ij}^{(hr)}\, dt\right)$$

But we know that for some x, say x_0, and for some i, j, say i_0, j_0, $u_{i_0 j_0}(x_0) = M$. Then since

$$\left|\sum_{h,\,r=1}^{n} \int_{\kappa_{h,r}}^{x_0} e^{\lambda \int_t^{x_0} \{\gamma_h(\xi)-\gamma_r(\xi)\}\,d\xi} \omega_{i_0 j_0}^{(hr)}(x_0, t)\, dt\right| < K\,M,$$

we have for the i_0, j_0 element, $M < \dfrac{KM}{\lambda^m}$,

i.e.
$$M\left(1 - \frac{K}{\lambda^m}\right) < 0 \, .$$

Inasmuch as M is positive and λ can be taken arbitrarily large this involves a contradiction, and proves the hypothesis untenable. Hence $|\rho_{ih}| \neq 0$. Accordingly there corresponds to every choice of $C \neq O$, and hence to the particular choice $C = I$, a unique $D \neq O$, and hence a solution $Y(x)$. By equation (44) the $U(x)$ for this $Y(x)$ must satisfy

$$U(x) = I - \frac{1}{\lambda^m}\, \psi(x, \lambda) \, .$$

We infer from this the inequality,

$$M < 1 + \frac{KM}{\lambda^m} \, ,$$

whence it follows that for λ sufficiently large, M is less that 2 and $\psi(x, \lambda)$ is uniformly bounded, for λ in the sector in question and further $|\lambda| > N$. Accordingly we have

$$Y(x) = \left(\delta_{ij} + \frac{\psi_{ij}(x, \lambda)}{\lambda^m}\right) (s_{ij}(x))\, E(x),$$

i.e.
$$Y(x) = \left(s_{ij}(x) + \frac{\nu_{ij}(x, \lambda)}{\lambda^m}\right) E(x).$$

Since the matrices $Y_0(x)$ and D are analytic in λ the same is true also of $Y(x)$. It is seen, moreover, that $|Y| \neq 0$. Hence, when the functions $\gamma_i(x)$ satisfy the conditions of the theorem on page 83 there exists in every sector of the type described above a matrix solution $Y(x)$ whose elements are continuous in x and analytic in λ, and are the same as those of $\bar{S}(x)$ to terms in $\frac{1}{\lambda^{m-1}}$. But by construction the elements of $\bar{S}(x)$ are the same as those of $S(x)$ to terms in $\frac{1}{\lambda^m}$. Hence we have the

Theorem: Given any formal solution $S(x)$ of equation (26) in which $R(x)$ and $B(x)$ both possess derivatives up to and including those of order k, then in each sector of the complex plane within which none of the quantities $R\{\lambda\{\gamma_h(x) - \gamma_r(x)\}\}$ change sign there exists an actual matrix solution $Y(x)$ which is continuous with its first derivative in x and analytic in λ, and is throughout the sector identical with $S(x)$ to terms in $\frac{1}{\lambda^k}$.

In virtue of this there exists in any sector of the type described a pair of actual associated matrix solutions of the form

$$(45) \qquad \left\{ \begin{aligned} Y(x) &= (e^{\lambda\Gamma_j(x)+B_j(x)}\, [\delta_{ij}]_{k-1}) \\ Z(x) &= (e^{-\lambda\Gamma_i(x)-B_i(x)}\, [\delta_{ij}]_{k-1}), \end{aligned} \right.$$

provided only that the matrices $R(x)$ and $B(x)$ are differentiable k terms. In all cases the functions $[\delta_{ij}]_{k-1}$ which occur in the expressions for $Y(x)$ and $Z(x)$ are, of course, respectively identical, to terms in $1/\lambda^{k-1}$ with the series $[\delta_{ij}]$ in the formulas (37) and (38).

In the course of the deduction of forms (45) it was assumed throughout that $n \geqq 2$. Direct integration of the equation, however, shows that in the case $n = 1$ there exist solutions which are of this form over the entire plane. Accordingly formulas (45) may be used in every case. The explicit forms

$$Y(x) = \left(e^{\lambda \Gamma_j(x)+B_j(x)}\left\{\delta_{ij} + \frac{\varphi_{ij}(x, \lambda)}{\lambda}\right\}\right),$$

$$Z(x) = \left(e^{-\lambda \Gamma_i(x)-B_i(x)}\left\{\delta_{ij} + \frac{\psi_{ij}(x, \lambda)}{\lambda}\right\}\right),$$

with φ_{ij}, ψ_{ij} bounded, hold if $R(x)$, $B(x)$ are differentiable.

Section VII.

The characteristic values of the system

$$Y'(x)\cdot = \{R(x)\lambda + B(x)\}\, Y(x)\cdot$$
$$W_a Y(a)\cdot + W_b Y(b)\cdot = O.$$

It was shown on page 65 that a necessary and sufficient condition that the vector system

$$(46) \qquad \begin{aligned} Y'\cdot &= \{R\lambda + B\}\, Y\cdot \\ W_a\, Y(a)\cdot + W_b\, Y(b)\cdot &= 0 \qquad (|\, W_a\,| \neq 0,\ |\, W_b\,| \neq 0), \end{aligned}$$

has a solution is that

$$(47) \qquad |\, W_a\, Y(a) + W_b\, Y(b)\,| = 0,$$

$Y(x)$ being any matrix solution of equation (26). Let us make the specific form of this condition in any sector of the λ plane apparent by substituting a $Y(x)$ which is analytic in λ for $|\,\lambda\,| > N$, and which has within the sector the form

$$(48) \qquad Y(x) = (e^{\lambda \Gamma_j(x)+B_j(x)}\,[\delta_{ij}]),$$

as determined in the preceding sections. Choosing the α of formula (20) as $\alpha = a$ it follows that $Y(a) = ([\delta_{ij}])$.

Substituting from (48) in the determinant on the left of equation (47), (call it $D(\lambda)$) we have

$$D(\lambda) = | W_a([\delta_{ij}]) + W_b(e^{\lambda\Gamma_j(b)+B_j(b)}[\delta_{ij}]) |,$$

i.e.

$$D(\lambda) = | [w_{rc}^{(a)}] + [w_{rc}^{(b)}] e^{\lambda\Gamma_c(b)+B_c(b)} | . \tag{49}$$

In this as in subsequent formulas when the expressions in question are determinants the letters r and c rather than i and j are used to indicate row and column respectively.

Due to the fact that those and only those values of λ which satisfy equation (47) are characteristic values, i.e. values which yield solutions of system (46), equation (47) is known as the *characteristic equation* of the system in question.

The introduction at this point of quantities δ_{ij}^{*} and δ_{ij}^{**} will do much toward simplifying the further discussion. These quantities are defined by the relations

$$\delta_{ij}^{*} \begin{cases} = \delta_{ij} \text{ when } R\{\lambda\Gamma_j(b)\} \leqq 0 \\ = 0 \text{ when } R\{\lambda\Gamma_j(b)\} > 0, \end{cases} \qquad \delta_{ij}^{**} \begin{cases} = 0 \text{ when } R\{\lambda\Gamma_j(b)\} \leqq 0 \\ = \delta_{ij} \text{ when } R\{\lambda\Gamma_j(b)\} > 0. \end{cases}$$

It is to be noted that δ_{jj}^{*} and δ_{jj}^{**} are functions of arg λ alone.

Suppose now that the rays[9] $R\{\lambda\Gamma_i(b)\} = 0$, $i = 1, 2, \ldots n$, are drawn in the plane of the parameter λ and let one of the sectors bounded by a pair of adjacent rays of this kind, $R\{\lambda\Gamma_k(b)\} = 0$, and $R\{\lambda\Gamma_l(b)\} = 0$, be denoted by σ_{kl}. Then if $\overset{*}{\delta}{}_{ij}^{(kl)}$ and $\overset{**}{\delta}{}_{ij}^{(kl)}$ are respectively the δ_{ij}^{*} and δ_{ij}^{**} for some (any) particular value of λ within σ_{kl}, we have the identities $\delta_{ij}^{*} \equiv \overset{*}{\delta}{}_{ij}^{(kl)}$, $\delta_{ij}^{**} \equiv \overset{**}{\delta}{}_{ij}^{(kl)}$, for all λ in the interior of the sector in question. It becomes apparent in virtue of the relations

$$[w_{rc}^{(a)}] + [w_{rc}^{(b)}] e^{\lambda\Gamma_c(b)+B_c(b)} = \begin{cases} [w_{rc}^{(a)}] \text{ when } R\{\lambda\Gamma_c(b)\} < 0 \\ [w_{rc}^{(b)}] e^{\lambda\Gamma_c(b)+B_c(b)} \\ \qquad \text{when } R\{\lambda\Gamma_c(b)\} > 0, \end{cases} \tag{50}$$

and the fact that $R\{\lambda\Gamma_i(b)\} \neq 0$, $i = 1, 2, \ldots n$, that $D(\lambda)$ takes, for any λ in the interior of such a sector the form

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[9] A half-line issuing from the origin of the λ plane will be referred to as a *ray*.

$$(51) \qquad D(\lambda) = |\, [w_{rc}^{(a)}]\, \overset{*}{\delta}_{cc} + [w_{rc}^{(b)}]\, \overset{**}{\delta}_{cc}\, e^{\lambda\Gamma_c(b) + B_c(b)}\, |.$$

Factoring from the determinant the exponential factors, any one of which occurs in each element of an entire column, the further alternative form

$$(52) \qquad D(\lambda) = \prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b) + B_k(b)\}\overset{**}{\delta}_{kk}}\, |\, [w_{rc}^{(a)}]\, \overset{*}{\delta}_{cc} + [w_{rc}^{(b)}]\, \overset{**}{\delta}_{cc}\, |$$

is obtained. It becomes necessary at this point to differentiate between certain types of conditions which inherently characterize any particular system of type (46).

The conditions of the system will be said to be *regular* (i) if $n = 1$, or if, when $n \geqq 2$, $\arg \Gamma_j(b) \neq \arg \{\pm \Gamma_i(b)\}$ when $j \neq i$, and (ii) the boundary conditions are such that each of the determinants

$$(53) \qquad W^{(kl)} = |\, w_{rc}^{(a)}\, \overset{*}{\delta}{}_{cc}^{(kl)} + w_{rc}^{(b)}\, \overset{**}{\delta}{}_{cc}^{(kl)}\, |$$

differs from zero.

The conditions of the system will be called *irregular* if either of these conditions is not satisfied. In the further discussion we shall consider first the case of regular conditions and then the particular type of irregular case which results from dropping the regularity condition (i) above.

Case I. Regular Conditions. In this case the rays $R\{\lambda\Gamma_i(b)\} = 0$, $i = 1, 2, \ldots n$, are distinct and divide the plane into $2n$ sectors of type σ_{kl}. Let us fix the attention upon any one of these rays, say the ray $R\{\lambda\Gamma_\nu(b)\} = 0$, and let $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ denote the abutting sectors, the former being that within which $R\{\lambda\Gamma_\nu(b)\} < 0$. For λ on the ray $R\{\lambda\Gamma_\nu(b)\} = 0$ the ν^{th} column of $D(\lambda)$ consists of terms whose form cannot be abbreviated by means of relations (50). Retaining therefore the original expressions for the elements of this column we have

$$D(\lambda) = |\, [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + [w_{rc}^{(b)}]\, \{\overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu}\}\, e^{\lambda\Gamma_c(b) + B_c(b)}\, |.$$

For λ either within $\sigma_{\mu\nu}$ or within $\sigma_{\nu\tau}$, on the other hand, the form of $D(\lambda)$ is given by formula (51), the difference in the value of $D(\lambda)$ in these sectors being accounted for by the difference in the values of $\overset{*}{\delta}_{cc}$ and $\overset{**}{\delta}_{cc}$, $c = 1, 2, \ldots, n$.

But it is readily seen that

$$(54) \qquad \overset{*}{\delta}{}^{(\nu\tau)}_{cc} = \overset{*}{\delta}{}^{(\mu\nu)}_{cc} - \delta_{c\nu}, \qquad \overset{**}{\delta}{}^{(\nu\tau)}_{cc} = \overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}.$$

Hence if S is any sector which includes the ray $R\{\lambda\Gamma_\nu(b)\} = 0$ and overlaps a part of each of the sectors $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ we have, for a λ in S, the following expressions

$$D(\lambda) = |\, [w^{(a)}_{rc}]\, \overset{*}{\delta}{}^{\mu\nu}_{cc} + [w^{(b)}_{rc}]\, \overset{**}{\delta}{}^{(\mu\nu)}_{cc}\, e^{\lambda\Gamma_c(b) + B_c(b)}|, \text{ for } \lambda \text{ within } \sigma_{\mu\nu},$$

$$D(\lambda) = |\, [w^{(a)}_{rc}]\, \overset{*}{\delta}{}^{(\mu\nu)}_{cc} + [w^{(b)}_{rc}]\, \{\overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}\} e^{\lambda\Gamma_c(b) + B_c(b)}|, \text{ for } \lambda \text{ on the ray } R\{\lambda\Gamma_\nu(b)\} = 0,$$

$$D(\lambda) = |\, [w^{(a)}_{rc}]\, \{\overset{*}{\delta}{}^{(\mu\nu)}_{cc} - \delta_{c\nu}\} + [w^{(b)}_{rc}]\, \{\overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}\}\, e^{\lambda\Gamma_c(b) + B_c(b)}|, \text{ for } \lambda \text{ within } \sigma_{\nu\tau},$$

It is readily seen from this that $D(\lambda)$ is given for any λ in S by the formula

$$(55) \qquad D(\lambda) = |\, [w^{(a)}_{rc}]\, \overset{*}{\delta}{}^{(\mu\nu)}_{cc} + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}\} e^{\lambda\Gamma_c(b) + B_c(b)}|.$$

Factoring from this determinant the product $\prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b) + B_k(b)\}\, \overset{**}{\delta}{}^{(\mu\nu)}_{kk}}$ we have

$$(56) \qquad D(\lambda) = \prod_{k=1}^{n} e^{\lambda\{\Gamma_k(b) + B_k(b)\}\, \overset{**}{\delta}{}^{(\mu\nu)}_{kk}}\, \overline{D}(\lambda),$$

where

$$\overline{D}(\lambda) = |\, [w^{(a)}_{rc}]\, \overset{*}{\delta}{}^{(\mu\nu)}_{cc} + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}\}\, e^{\lambda\Gamma_o(b) + B_\nu(b)}\, |.$$

Now $\overline{D}(\lambda)$ is seen to have elements consisting of a single term in every column but the ν^{th}, the elements of that column being binomial. Consequently the expression of the determinant as the sum of two others is possible, i.e.

$$\overline{D}(\lambda = |\, [w^{(a)}_{rc}]\, \overset{*}{\delta}{}^{(\mu\nu)}_{cc} + [w^{(b)}_{rc}]\, \overset{**}{\delta}{}^{(\mu\nu)}_{cc}\, | + |\, [w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\mu\nu)}_{cc} - \delta_{c\nu}\} + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\mu\nu)}_{cc} + \delta_{c\nu}\}|e^{\lambda\Gamma_\nu(b) + B_\nu(b)}$$

or, in view of relations (53) and (54),

$$(57) \qquad \overline{D}(\lambda) = [W^{\mu\nu}] + [W^{\nu\tau}]\, e^{\Gamma_\nu\lambda(b) + B_\nu(b)}.$$

Since the roots of $D(\lambda) = 0$ and those of $\overline{D}(\lambda) = 0$ are the same we have, therefore, as the characteristic equation

$$(58) \qquad [W^{(\mu\nu)}] + [W^{(\nu\tau)}]\, e^{\lambda\Gamma_\nu(b)+B_\nu(b)} = 0.$$

This yields, since $W^{(\mu\nu)} \neq 0$, the equation

$$e^{\lambda\Gamma_\nu(b)+B_\nu(b)} = \left[-\frac{W^{(\mu\nu)}}{W^{(\nu\tau)}}\right] = \frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + \epsilon,$$

where ϵ is here introduced as a generic symbol for functions which approach the limit zero uniformly as $|\lambda|$ increases beyond limit.

Solving for λ, and observing that $\log(K+\epsilon) = \log K + \epsilon$, we see, therefore, that every characteristic value which lies in sector S is of the form

$$(59) \quad \lambda_p = \frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + \epsilon(\lambda_p) + 2p\pi i\right\},$$

where p is a positive integer. Moreover, since $Y(x)$ is analytic in λ throughout the sector S, $|\lambda| > N$, the same is readily seen to be true of ϵ also.

Consider now a small circle of fixed radius r drawn about the point

$$\frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + 2p\pi i\right\},$$

for any given p. Then for a proper choice of origin in the λ plane (see page 98) this circle lies entirely within the sector $\sigma_{\mu\nu}$ or $\sigma_{\nu\tau}$, and $\left|\frac{\epsilon}{\Gamma_\nu(b)}\right| < r$ for $|\lambda| > N$. Also if p is sufficiently large the point

$$\frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + \epsilon + 2p\pi i\right\}$$

lies within the circle for all λ on the circumference. Consequently

$$\arg\left\{\lambda - \frac{1}{\Gamma_\nu(b)}\left\{-B_\nu(b) + \log\frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + \epsilon + 2p\pi i\right\}\right\}$$

increases by 2π as λ describes this circumference, and just one root of equation (59) is accordingly seen to lie within the circle. Since it is

readily verified that every such root is a characteristic value it is seen that (59) determines such a value for every p which is sufficiently large, i.e. that for large values of λ the characteristic values lie approximately along a line parallel to the ray $R\{\lambda\Gamma_\nu(b)\} = 0$, the distance between two adjacent ones approaching as a limit the finite length $2\pi / |\Gamma_\nu(b)|$, as $|\lambda|$ increases indefinitely. From the derivation of this result it is seen moreover, that a similar sequence of characteristic values lies near each ray $R\{\lambda\Gamma_i(b)\} = 0$, and that no further distribution of characteristic values exists.

Case II. A Type of Irregular Conditions.[10] Let us suppose now that the regularity condition (ii) is fulfilled, that $n \geqq 2$, but that

$$\arg \Gamma_i(b) = \arg \{\pm\Gamma_j(b)\} \tag{60}$$

for the pair of values $i = \nu_1$, $j = \nu_2$. We have then a case of irregular conditions, and while we shall restrict the discussion to the case when the relation (60) holds for only a single set of values i, j, the reasoning to be employed is typical and may be applied with equal success to the cases in which a greater number of the points $\Gamma_i(b)$ are collinear with the origin of the complex plane. It is only for the sake of brevity that the simplest, rather than the most general case which results from dropping the regnlarity condition (i) is treated.

A review of the discussion applied to the case of regular conditions readily shows that the methods employed there apply equally well to the case in hand and yield the same results in any sector of the λ plane which does not contain the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$. It is, therefore, necessary to consider here only the distribution of characteristic values in a sector S containing this line. Along the line in question neither the expressions for the elements of the ν_1^{th} nor of the ν_2^{th} column of $D(\lambda)$ can be contracted by means of the relations (50). Accordingly it is found that $D(\lambda)$ takes the form

$$D(\lambda) = \left| [w_{rc}^{(a)}]\, \overset{*}{\delta}{}_{cc}^{(\nu)} + [w_{rc}^{(b)}]\, \{\overset{**}{\delta}{}_{cc}^{(\nu)} + \delta_{c\nu_1} + \delta_{c\nu_2}\}\, e^{\lambda\Gamma_c(b) + B_c(b)} \right| \tag{61}$$

throughout the entire sector S, $\overset{*}{\delta}{}_{cc}^{(\nu)}$ and $\overset{**}{\delta}{}_{cc}^{(\nu)}$ representing the quantities δ^*_{cc} and δ^{**}_{cc} for λ on the ray $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$ which lies in S. Factoring

[10] For the discussion of a differential system representing a different type of irregular conditions see Hopkins, loc. cit.

from (61) the product $\prod\limits_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\,\overset{**}{\delta}{}^{(\nu)}_{kk}}$ we have

$$D(\lambda) = \prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\,\overset{**}{\delta}{}^{(\nu)}_{kk}}\,\overline{\overline{D}}(\lambda), \tag{62}$$

where

$$\overline{\overline{D}}(\lambda) = \Big|\,[w^{(a)}_{rc}]\overset{*}{\delta}{}^{(\nu)}_{cc} + [w^{(b)}_{rc}]\big\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1}\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + \delta_{c\nu_2}\, e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)}\big\}\Big|.$$

The determinant $\overline{\overline{D}}(\lambda)$ is seen, therefore, to have monomial elements in each column except the ν_1^{th} and the ν_2^{th}, the elements in these columns being binomials. It is clear, therefore, that $\overline{\overline{D}}(\lambda)$ can be expressed as the sum of four determinants each containing only monomial elements, i.e.

$$\begin{aligned}\overline{\overline{D}}(\lambda) = {} & \big|\,[w^{(a)}_{rc}]\,\overset{*}{\delta}{}^{(\nu)}_{cc} + [w^{(b)}_{rc}]\,\overset{**}{\delta}{}^{(\nu)}_{cc}\,\big| + \big|\,[w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_1}\} \\ & + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1}\}\big|\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + \\ & \big|\,[w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_2}\} + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_2}\}\big|\, e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} + \\ & \big|\,[w^{(a)}_{rc}]\{\overset{*}{\delta}{}^{(\nu)}_{cc} - \delta_{c\nu_1} - \delta_{c\nu_2}\} \\ & + [w^{(b)}_{rc}]\{\overset{**}{\delta}{}^{(\nu)}_{cc} + \delta_{c\nu_1} + \delta_{c\nu_2}\}\,\big|\, e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)}\,.\end{aligned} \tag{63}$$

Let us again denote by $\sigma_{\mu\nu}$ and $\sigma_{\nu\tau}$ the sectors abutting on the ray $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$ in which $R\{\lambda\Gamma_{\nu_1}(b)\} < 0$ and > 0 respectively. It is necessary to consider the case in which $\arg\Gamma_{\nu_1}(b) = \arg\Gamma_{\nu_2}(b)$ and that in which $\arg\Gamma_{\nu_1}(b) = \arg\{-\Gamma_{\nu_2}(b)\}$.

Sub-case A. $\qquad \arg\Gamma_{\nu_1}(b) = \arg\Gamma_{\nu_2}(b)$.

In this case the quantities $R\{\lambda\Gamma_{\nu_1}(b)\}$ and $R\{\lambda\Gamma_{\nu_2}(b)\}$ are of the same sign throughout the sector S, and the relations

$$\left\{\begin{aligned} \overset{*}{\delta}{}^{(\nu\tau)}_{cc} + \delta_{c\nu_1} + \delta_{c\nu_2} &= \overset{*}{\delta}{}^{(\nu)}_{cc} = \overset{*}{\delta}{}^{(\mu\nu)}_{cc} \\ \overset{**}{\delta}{}^{(\nu\tau)}_{cc} - \delta_{c\nu_1} - \delta_{c\nu_2} &= \overset{**}{\delta}{}^{(\nu)}_{cc} = \overset{**}{\delta}{}^{(\mu\nu)}_{cc} \end{aligned}\right. \tag{64}$$

are readily verified. Defining the determinants W_1 and W_2 by the formulas

$$
(65) \qquad
\begin{aligned}
W_1 &= | w_{rc}^{(a)} \{ \overset{*}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu_1} \} + w_{rc}^{(b)} \{ \overset{**}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu_1} \} | \\
W_2 &= | w_{rc}^{(a)} \{ \overset{*}{\delta}{}_{cc}^{(\nu\tau)} + \delta_{c\nu_1} \} + w_{rc}^{(b)} \{ \overset{**}{\delta}{}_{cc}^{(\nu\tau)} - \delta_{c\nu_1} \} |
\end{aligned}
$$

we have $\overline{\overline{D}}(\lambda) = \overline{D}(\lambda)$ where, in view of (64), $\overline{D}(\lambda)$ satisfies the equation (56) and is given by

$$
(66) \quad \overline{D}(\lambda) = [W^{(\mu\nu)}] + [W_1]\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W_2]\, e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} \\
+ [W^{(\nu\tau)}]\, e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)}
$$

Sub-case B. $\qquad \arg \Gamma_{\nu_1}(b) = \arg \{ - \Gamma_{\nu_2}(b) \}$.

In this case $R\{\lambda\Gamma_{\nu_2}(b)\}$ has throughout S the sign opposite to that of $R\{\lambda\Gamma_{\nu_1}(b)\}$, and it is found that

$$
(67) \qquad
\begin{aligned}
\overset{*}{\delta}{}_{cc}^{(\nu\tau)} + \delta_{c\nu_1} &= \overset{*}{\delta}{}_{cc}^{(\nu)} = \overset{*}{\delta}{}_{cc}^{(\mu\nu)} + \delta_{c\nu_2} \\
\overset{**}{\delta}{}_{cc}^{(\nu\tau)} - \delta_{c\nu_1} &= \overset{**}{\delta}{}_{cc}^{(\nu)} = \overset{**}{\delta}{}_{cc}^{(\mu\nu)} - \delta_{c\nu_2}\,.
\end{aligned}
$$

Accordingly we have

$$
\overline{\overline{D}}(\lambda) = [W_2] + [W^{(\nu\tau)}]e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W^{(\mu\nu)}]e^{\lambda\Gamma_{\nu_2}(b)+B_{\nu_2}(b)} \\
+ [W_1]e^{\lambda\{\Gamma_{\nu_1}(b)+\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)+B_{\nu_2}(b)}.
$$

If now we define $\overline{D}(\lambda)$ by the relation

$$
\overline{\overline{D}}(\lambda)\, e^{-\lambda\Gamma_{\nu_2}(b)-B_{\nu_2}(b)} = \overline{D}(\lambda),
$$

it is readily verified on the basis of formulas (67) that $\overline{D}(\lambda)$ again satisfies equation (56) while it is given in this case by

$$
(68) \quad \overline{D}(\lambda) = [W^{(\mu\nu)}] + [W_1]\, e^{\lambda\Gamma_{\nu_1}(b)+B_{\nu_1}(b)} + [W_2]e^{-\lambda\Gamma_{\nu_2}(b)-B_{\nu_2}(b)} \\
+ [W^{(\nu\tau)}]e^{\lambda\{\Gamma_{\nu_1}(b)-\Gamma_{\nu_2}(b)\}+B_{\nu_1}(b)-B_{\nu_2}(b)}.
$$

Let us suppose now that the notation has been so chosen that $| \Gamma_{\nu_2}(b) | \geqq | \Gamma_{\nu_1}(b) |$, and set $\left| \dfrac{\Gamma_{\nu_2}(b)}{\Gamma_{\nu_1}(b)} \right| = r$. Then upon dividing equations (66) and (68) through by $[W^{(\mu\nu)}]$, (recall that $W^{(\mu\nu)} \neq 0$ by

hypothesis) it is seen that the characteristic equation, i.e. $\bar{D}(\lambda) = 0$, is in each case of the form

$$(69) \qquad 1 + [c_1]e^{\lambda\Gamma_{\nu_1}(b)} + [c_2]\, e^{r\lambda\Gamma_{\nu_1}(b)} + [c_3]\, e^{\{1+r\}\lambda\Gamma_{\nu_1}(b)} = 0,$$

where c_i, $i = 1, 2, 3$, are complex constants, $c_3 \neq 0$ by hypothesis, and r is a real constant $r \geqq 1$.

Wilder [11] has shown that the roots of this equation are asymptotically represented by those of the equation

$$(70) \qquad 1 + c_1\, e^{\lambda\Gamma_{\nu_1}(b)} + c_2\, e^{r\lambda\Gamma_{\nu_1}(b)} + c_3\, e^{\{1+r\}\lambda\Gamma_{\nu_1}(b)} = 0,$$

and has discussed the distribution of the roots of this equation. We shall proceed to this discussion, observing, however, first that when $| \Gamma_{\nu_2}(b) |$ and $| \Gamma_{\nu_1}(b) |$ are commensurable a far simpler treatment is possible. In that case $r = \dfrac{p}{q}$ where p and q are integers and the equation (70) is an algebraic equation of degree $(p + q)$ in $e^{\frac{\lambda\Gamma_{\nu_1}(b)}{q}}$. Accordingly it has $(p + q)$ roots, i.e.

$$e^{\frac{\lambda\Gamma_{\nu_1}(b)}{q}} = \alpha_j, \qquad j = 1, 2, \ldots, (p + q)$$

from which it follows that

$$(71) \qquad \lambda_k = \frac{q}{\Gamma_{\nu_1}(b)} \{\log \alpha_j + 2k\pi i\}.$$

In this case, therefore, the characteristic values which lie in sector S for $| \lambda | > N$ are asymptotically represented by a set of points which are spaced at intervals of length $\dfrac{2q\pi}{| \Gamma_{\nu_1}(b) |}$ on $(p + q)$ lines (not necessarily distinct) parallel to the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$.

When $\Gamma_{\nu_2}(b)$ and $\Gamma_{\nu_1}(b)$ are incommensurable no such simple treatment is possible. The distribution of the characteristic values may be obtained, however, by Wilder's procedure,[12] which follows.

Setting $\lambda\Gamma_{\nu_1}(b) = z = x + iy$ we have as the equation (70)

$$f(z) = 1 + c_1 e^{z} + c_2 e^{rz} + c_3 e^{\{1+r\}z} = 0,$$

---

11 Wilder, loc. cit., p. 423.
12 Cf. Wilder, loc. cit., pp. 420–422.

and it is readily verified that there corresponds to each choice of an arbitrarily small positive constant χ, some value of x, say $x = X$, such that

$$(72)\qquad \begin{cases} \text{(i)} \quad |1 - f(-z)| < \chi, \\ \text{(ii)} \quad \left| c_3 - \dfrac{f(z)}{e^{\{1+r\}z}} \right| < \chi, \text{ for } x \leqq X. \end{cases}$$

We shall assume that χ is chosen sufficiently small to preclude the vanishing of $f(z)$ outside or on the boundary of the region $|x| \leqq X$.

Now $f(z)$ is analytic throughout the entire finite plane, and hence it is possible to find in any interval of the Y axis, however small, some point $y = y_0$ which is such that the line $y = y_0$ contains no zero of $f(z)$. Let $y = Y_1$ and $y = Y_2$ be any two such lines and consider the rectangle K bounded by them and the lines $x = X$ and $x = -X$. We shall determine the number of zeros of $f(z)$ within K by observing the increase in $\arg f(z)$ as z describes the perimeter.

We have $\arg f(z) = \sin^{-1} \dfrac{I\{f(z)\}}{|f(z)|}$, where $I\{f(z)\}$ denotes the coefficient of $\sqrt{-1}$ in the expression for $f(z)$. Moreover, for $y =$ constant $I\{f(z)\}$ has the form

$$I\{f(z)\} = d_1 e^x + d_2 e^{rx} + d_3 e^{\{1+r\}x},$$

where the coefficients, d_i, $i = 1, 2, 3$ are real constants. The finite zeros of $I\{f(z)\}$, and hence those of $\dfrac{I\{f(z)\}}{|f(z)|}$, being the roots of the equation

$$d_1 + d_2 e^{\{r-1\}x} + d_3 e^{rx} = 0,$$

are, however, separated by the finite zeros of the derivative of the left-hand member, namely by the roots of the equation

$$d_2\{r-1\} + d_3 r e^x = 0.$$

Since this equation is satisfied by at most one value of x it follows that $\dfrac{I\{f(z)\}}{|f(z)|}$ vanishes at most twice, and consequently that $\arg f(z)$ changes

by less than 3π as x varies between the limits $x = -X$ and $x = +X$ along a line $y =$ constant.

Because of the relation (72i), however, we know that for every z on line $x = -X$, $f(z)$ lies within a circle of radius χ about the point $z = 1$, and hence that as z moves along this line $\arg f(z)$ changes by less than $2\sin^{-1}\chi$. Similarly relation (72ii) shows that $\arg\{f(z)/e^{(1+r)z}\}$ changes by less than $2\sin^{-1}\chi$ as z moves along the line $x = X$. From the identity

$$\arg f(z) = \arg e^{\{1+r\}z} + \arg \frac{f(z)}{e^{\{1+r\}z}}$$

it follows, therefore, that the increase in $\arg f(z)$ as z moves along the line $x = X$ from $y = Y_1$ to $y = Y_2$ lies between $(1+r)\{Y_2 - Y_1\} + 2\sin^{-1}\chi$ and $(1+r)\{Y_2 - Y_1\} - 2\sin^{-1}\chi$. Consequently the increase in $\arg f(z)$ as z describes the perimeter of K lies between

$(1+r)\{Y_2 - Y_1\} + 6\pi + 4\sin^{-1}\chi$ and $(1+r)\{Y_2 - Y_1\} - 6\pi - 4\sin^{-1}\chi$,

and accordingly the number of zeros located in the interior of K must lie between

$$\frac{(1+r)\{Y_2 - Y_1\}}{2\pi} + 3 + \frac{2}{\pi}\sin^{-1}\chi$$

and

$$\frac{(1+r)\{Y_2 - Y_1\}}{2\pi} - 3 - \frac{2}{\pi}\sin^{-1}\chi.$$

Since χ can be chosen arbitrarily small, however, this means that the number of characteristic values between any two lines $y = c_1$ and $y = c_1 + l$ is at least $\frac{1+r}{2\pi} l - 3$, and cannot on the other hand exceed $\frac{1+r}{2\pi} l + 3$.

Summarizing the results it is seen, therefore, that the characteristic values in sector S lie in a strip bounded by two parallels to the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$, and that they are so distributed throughout this strip that for $|\lambda| > N$ no more than three lie between any two lines which are at a distance $d < \frac{2\pi}{1+r}$ from each other and are perpendicular to the line $R\{\lambda\Gamma_{\nu_1}(b)\} = 0$.

Section VIII.

The formal expansion of an arbitrary vector.

It was shown in the preceding section that both under regular conditions and under the type of irregular conditions discussed the characteristic values for system (46) are numerable and cluster about the point $\lambda = \infty$. Denoting these values by $\lambda_1, \lambda_2, \lambda_3, \ldots$ it is possible, therefore, to assign the subscripts in such manner that $|\lambda_m| \leqq |\lambda_{m+1}|$. Moreover then $\lim\limits_{m=\infty} |\lambda_m| = \infty$.

Assuming that system (46) is simply compatible at the characteristic values there exists for each of these values just one solution of the system in question and just one solution of its adjoint system. These solutions for $\lambda = \lambda_k$ will be designated by $Y^{(k)}(x)\cdot$ and $\cdot Z^{(k)}(x)$ respectively.

Now if $\lambda = 0$ is a characteristic value for system (46) let the parameter be changed by setting $\lambda = \bar{\lambda} + c$, c being a constant. Equation (46) then becomes

$$Y'(x)\cdot = \{ R(x)\,\bar{\lambda} + \bar{B}(x)\}\, Y(x)\cdot,$$

where

$$\bar{B}(x) = c\,R(x) + B(x).$$

The characteristic values of the system thus modified are $\bar{\lambda} = \lambda_k - c$ and it is clearly always possible to choose c so that $\bar{\lambda} = 0$ is not a characteristic value. No loss of generality is entailed, therefore, by the assumption, which will be made, that $\lambda_k \neq 0$ for any k.

Writing the equation (46) for $\lambda = \lambda_k$ in the form

$$Y^{(k)\prime}(x)\cdot = B(x)\,Y^{(k)}(x)\cdot + \lambda_k\,R(x)\,Y^{(k)}(x)\cdot$$

and considering this as a non-homogeneous equation we have from page 67

$$Y^{(k)}(x)\cdot = \lambda_k \int_a^b G(x,t)\,R(t)\,Y^{(k)}(t)\cdot dt, \tag{73}$$

where $G(x, t)$ is the Green's function for the system

$$\begin{aligned} Y'(x)\cdot &= B(x)\, Y(x)\cdot \\ W_a\, Y(a)\cdot + W_b\, Y(b)\cdot &= 0.\text{[13]} \end{aligned} \tag{74}$$

In precisely similar manner we have from the adjoint system

$$\cdot Z^{(l)}(x) = -\lambda_l \int_a^b \cdot Z^{(l)}(t)\, R(t)\, H(x, t)\, dt,$$

or, substituting from the relation

$$G(t, x) \equiv -H(x, t).$$

$$\cdot Z^{(l)}(t) = \lambda_l \int_a^b \cdot Z^{(l)}(x)\, R(x)\, G(x, t)\, dx. \tag{75}$$

Consider, now, the integral

$$J = \int_a^b \int_a^b \cdot Z^{(l)}(x)\, R(x)\, G(x, t)\, R(t)\, Y^{(k)}(t)\cdot dx\, dt.$$

In view of relation (73) we have

$$\lambda_k J = \int_a^b \cdot Z^{(l)}(x)\, R(x)\, Y^{(k)}(x)\cdot dx,$$

while it is seen from (75) that

$$\lambda_l J = \int_a^b \cdot Z^{(l)}(t)\, R(t)\, Y^{(k)}(t)\cdot dt.$$

By subtraction, then

$$(\lambda_k - \lambda_l)\, J = 0,$$

and it follows that $J = 0$ provided $k \neq l$. But then $\lambda_l J = 0$, i.e.

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, Y^{(k)}(x)\cdot dx = 0, \text{ for } k \neq l. \tag{76}$$

---

[13] Since $\lambda = 0$ is not a characteristic value of system (46) system (74) is incompatible and the Green's function exists.

It can, moreover, be easily shown (see page 107) that when the system (46) is simply compatible at every characteristic value, this is not true for $k=l$, i.e.

$$\int_a^b \cdot Z^{(l)}(x)\; R(x)\; Y^{(l)}(x) \cdot dx \neq 0, \text{ for any } l.$$

Let us suppose now that an arbitrarily chosen vector $F(x)\cdot$ can be developed into a series of the form

$$F(x)\cdot = \sum_{k=1}^{\infty} c_k\, Y^{(k)}(x)\cdot\,. \tag{77}$$

Multiplying both sides of this equation on the left by the vector $\cdot Z^{(l)}(x)\; R(x)$ and integrating term by term we have formally

$$\int_a^b \cdot Z^{(l)}(x)\; R(x)\; F(x) \cdot dx = \sum_{k=1}^{\infty} c_k \int_a^b \cdot Z^{(l)}(x)\; R(x)\; Y^{(k)}(x) \cdot dx,$$

which in view of relation (76) reduces to

$$\int_a^b \cdot Z^{(l)}(x)\; R(x)\; F(x) \cdot dx = c_l \int_a^b \cdot Z^{(l)}(x)\; R(x)\; Y^{(l)}(x) \cdot dx. \tag{78}$$

Inasmuch as the matrix on each side of this equation is one all of whose elements are identical the equation may equally well be written

$$\left\{ \int_a^b \sum_{h=1}^{n} z_h^{(l)}(x)\; \gamma_h(x)\, f_h(x)\; dx \right\} (1) = \\ c_l \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(l)}\; (x)\; \gamma_h\; (x)\; y_h^{(l)}(x)\; dx \right\} (1),$$

whence

$$c_l = \frac{\displaystyle\int_a^b \sum_{h=1}^{n} z_h^{(l)}(x)\; \gamma_h(x)\, f_h(x)\; dx}{\displaystyle\int_a^b \sum_{h=1}^{n} z_h^{(l)}(x)\; \gamma_h(x)\; y_h^{(l)}(x)\; dx}.$$

Consequently we have

$$(79)\qquad F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\displaystyle\int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\,\gamma_h(x)\,f_h(x)\,dx}{\displaystyle\int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\,\gamma_h(x)\,y_h^{(k)}(x)\,dx} \right\} Y^{(k)}(x)\cdot$$

If, therefore, $F(x)\cdot$ may be developed into a series of form (74) which converges in such a manner as to legitimatize the processes above, then (79) is a necessary form for the series in question.

Thus far it has been stipulated only that $Y^{(k)}(x)\cdot$ and $\cdot Z^{(k)}(x)$ be respectively solutions of system (46) and its adjoint for $\lambda = \lambda_k$. But each of these systems is homogeneous, and if $\overline{Y}^{(k)}(x)\cdot$ and $\cdot\overline{Z}^{(k)}(x)$ are any particular solutions then $c\,\overline{Y}^{(k)}(x)\cdot$ and $\bar{c}\cdot\overline{Z}^{(k)}(x)$ are also solutions. Having chosen a definite pair $\overline{Y}^{(k)}(x)\cdot$ and $\cdot\overline{Z}^{(k)}(x)$ we have then

$$F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\displaystyle\int_a^b \sum_{h=1}^{n} \bar{c}\,\bar{z}_h^{(k)}(x)\,\gamma_h(x)\,f_h(x)\,dx}{\displaystyle\int_a^b \sum_{h=1}^{n} \bar{c}\,\bar{z}_h^{(k)}(x)\,\gamma_h(x)\,\bar{y}_h^{(k)}(x)\,dx} \right\} \overline{Y}^{(k)}(x)\cdot$$

If in particular $\bar{c}$ is chosen so that

$$\bar{c}\int_a^b \sum_{h=1}^{n} \bar{z}_h^{(k)}(x)\,\gamma_h(x)\,\bar{y}_h^{(k)}(x)\,dx = 1,$$

and the vector $\bar{c}\cdot\overline{Z}^{(k)}$, for this value of $\bar{c}$ is associated with $\overline{Y}^{(k)}\cdot$ so that the choice of one implies the choice of the other, we have, on dropping the bars over the letters,

$$(80)\qquad F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\,\gamma_h(x)\,f_h(x)\,dx \right\} Y^{(k)}(x)\cdot.$$

In using this formula it must be remembered that $\cdot Z^{(k)}(x)$ is determined as soon as the particular $Y^{(k)}(x)\cdot$ is chosen.

In a precisely analogous manner it may be found that a development of the type

$$\cdot \bar{F}(x) = \sum_{k=1}^{\infty} \bar{c}_k \cdot Z^{(k)}(x),$$

which converges in such manner that it may be integrated term by term after being multiplied by any of the vectors $R(x)\ Y^{(l)}(x)\cdot$, must necessarily coincide with the expansion

$$\cdot \bar{F}(x) = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} \bar{f}_h(x)\, \gamma_h\,(x)\, y_h^{(k)}(x)\, dx \right\} \cdot Z^{(k)}(x).$$

Inasmuch as the matrix $G(x, t)$ is not a vector the methods outlined above do not apply directly to the problem of expanding the Green's function. We shall proceed, therefore, as follows.

Let $G_j(x, t)\cdot$ denote the vector each of whose columns is the j^{th} column of $G(x, t)$. From the relation (80), $F(x)$ being replaced by $G_j(x,t)\cdot$, it follows that for any value of t, $G_j(x, t)\cdot$ can be formally developed, the series obtained being

$$G_j(x, t)\cdot \ = \sum_{k=1}^{\infty} \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(k)}(x)\, \gamma_h(x)\, g_{hj}\,(x, t)\, dx \right\} Y^{(k)}(x)\cdot. \tag{81}$$

In view of equation (75) written in the form

$$(z_j^{(l)}(t)) = \lambda_l \int_a^b \sum_{h=1}^{n} z_h^{(l)}(x)\, \gamma_h(x)\, g_{hj}\,(x, t)\, dx.$$

(81) reduces to

$$G_j(x, t)\cdot \ = \sum_{k=1}^{\infty} \frac{z_j^{(k)}(t)}{\lambda_k}\ Y^{(k)}(x)\cdot,$$

from which we obtain the relation

$$g_{ij}(x, t) = \sum_{k=1}^{\infty} \frac{z_j^{(k)}(t)\, y_i^{(k)}(x)}{\lambda_k}. \tag{82}$$

But since $\cdot Z^{(k)}(t)$ and $Y^{(k)}(x)\cdot$ are vectors

$$Y^{(k)}(x)\cdot\cdot Z^{(k)}(t) = \left(\sum_{l=1}^{n} y_i^{(k)}(x)\, z_j^{(k)}(t)\right) = n\,(y_i^{(k)}(x)\, z_j^{(k)}(t)).$$

Hence we have from (81)

$$G(x, t) = \sum_{k=1}^{\infty} \frac{Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)}{n\lambda_k}. \tag{83}$$

Having obtained this formal development of the Green's function it is possible to state the

THEOREM: If the development (83) converges to $G(x, t)$ in such a manner that a uniformly convergent series is obtained by multiplying (83) on the right by any matrix of continuous functions and integrating term by term, then any vector $F(x)\cdot$ the elements of which are continuous and have continuous first derivatives, and which satisfies the boundary conditions

$$W_aF(a)\cdot + W_bF(b)\cdot = O$$

is represented by a convergent development of the type (80).

Proof: The relation

$$F'(x)\cdot = B(x)\, F(x)\cdot + C(x)\cdot$$

defines the vector $C(x)\cdot$, and inasmuch as $F(x)\cdot$ is then a solution of the non-homogeneous system

$$Y'(x)\cdot = B(x)\, Y(x)\cdot + C(x)\cdot$$
$$W_aY(a)\cdot + W_bY(b)\cdot = O,$$

it is given by the formula

$$F(x)\cdot = \int_a^b G(x, t)\, C(t)\cdot dt$$ (see page 20).

Substituting for $G(x, t)$ its series development we have

$$F(x)\cdot = \int_a^b \sum_{k=1}^{\infty} \frac{Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)\, C(t)\cdot}{n\lambda_k}\, dt,$$

or, integrating term by term,

$$(84)\qquad F(x)\cdot = \sum_{k=1}^{\infty} Y^{(k)}(x)\cdot \left\{ \frac{1}{n\lambda_k} \int_a^b \cdot Z^{(k)}(t)\, C(t)\cdot dt \right\}\cdot$$

Now given any matrix of type $D\cdot$, and another of type $\cdot A\cdot$, we have $D\cdot\cdot A\cdot = \left(\sum_{l=1}^{n} d_i\, a\right) = (nad_i)$, i.e.

$$(85)\qquad D\cdot\cdot A\cdot = n\, a\, D\cdot$$

Hence (84) reduces to

$$F(x)\cdot = \sum_{k=1}^{\infty} \left\{ \frac{\int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, c_h(t)\, dt}{\lambda_k} \right\} Y^{(k)}(x)\cdot,$$

the series on the right converging uniformly to $F(x)\cdot$ Q. E. D.

In formula (12) the explicit expression for the Green's function $G(x, t)$ for an incompatible system of type (6) is given, Y_h and Z_h being respectively a solution of the differential matrix equation $Y'(x) = A(x)\, Y(x)$, and of its adjoint. Now for any λ, not a characteristic value, system (46) being incompatible is precisely of type (6). Consequently we have from (12)

$$(86)\qquad G(x, t, \lambda) = \overline{G}(x, t, \lambda) - Y(x, \lambda)\Delta^{-1}(\lambda)\{W_a\overline{G}(a, t, \lambda) + W_b\overline{G}(b, t, \lambda\},$$

for any λ not a characteristic value.

Let us investigate now the functional dependence of $G(x, t, \lambda)$ upon the parameter λ, choosing as the solutions $Y(x, \lambda)$ and $Z(t, \lambda)$ a pair which are analytic in λ. Recalling that when the determinant of a matrix A is denoted by A then $A^{-1} = \left(\frac{A_{ji}}{A}\right)$, where A_{ij} is the cofactor of the ij^{th} element of A, the explicit form of $\Delta^{-1}(\lambda)$ is seen to be

$$\Delta^{-1}(\lambda) = \left(\frac{D_{ji}(\lambda)}{D(\lambda)}\right).^{14}$$

---

14 This notation holds when $n = 1$ provided it is agreed that when A is a matrix of one element then $A_{11} \equiv 1$.

Now it was seen (page 87) that at a characteristic value $D(\lambda)$ vanishes. Let us assume further that the characteristic values of system (46) are all simple so that $D(\lambda)$ vanishes at each of these values to only the first order. Due to the relation

$$|D_{ij}| = D^{n-1}, \tag{87}$$

which is familiar from the theory of determinants,[15] it follows, then, that $D_{ij}(\lambda_k)$ cannot vanish for all i and j, since in the alternative case the left-hand side of (87) would vanish to the order n, while the right-hand side vanishes only to the order $(n-1)$. Hence $(D_{ji}(\lambda_k)) \neq O$, and $\Delta^{-1}(\lambda)$ has a simple pole at each of the characteristic values. Moreover, system (46) is simply compatible at the characteristic values as was assumed above. It is easily verified (i), that the poles of $\Delta^{-1}(\lambda)$ actually persist in $G(x, t, \lambda)$, and (ii), that $G(x, t, \lambda)$ has no other singularities.

Denoting by $G^{(k)}(x, t)$ the residue of $G(x, t, \lambda)$ at $\lambda = \lambda_k$ we obtain from (86) by familiar methods

$$G^k(x, t) = \frac{Y^{(k)}(x)\,(D_{ji}(\lambda_k))\,\{W_a Y^{(k)}(a) - W_b Y^{(k)}(b)\} Z^k(t)}{\left.\frac{d}{d\lambda} D(\lambda)\right]_{\lambda=\lambda_k}},$$

which, inasmuch as $\left.\frac{d}{d\lambda} D(\lambda)\right]_{\lambda=\lambda_k} \neq 0$, is of the form

$$G^{(k)}(x, t) = Y^{(k)}(x)\, C^{(k)}\, Z^{(k)}(t). \tag{88}$$

From this it follows that for fixed t, $G^{(k)}(x, t)$ is a solution of the system

$$\begin{cases} Y'(x) = \{R(x)\,\lambda_k + B(x)\}\; Y(x) \\ W_a Y(a) + W_b Y(b) = O. \end{cases} \tag{89}$$

But if $\overline{Y}(x)$ is a particular solution of the differential equation of this system every solution is of the form

$$Y(x) = \overline{Y}(x)\, C.$$

Substituting in the boundary conditions as on page 64 and denoting by (d_{ij}) the Δ which results from $Y = \overline{Y}$ we have

[15] Cf. Bôcher, loc. cit., p. 33.

(90) $$(d_{ij})\,(c_{ij}) = O,$$

or $$\sum_{l=1}^{n} d_{il}\, c_{lj} = 0, \qquad i, j, = 1, 2, \ldots n.$$

Now it was shown that $(D_{ij}(\lambda_k)) \neq O$, i.e. that the rank of $D(\lambda_k)$ is $(n-1)$. It follows that the solution of the system of linear equations

(91) $$\sum_{l=1}^{n} d_{il}\, c_l = 0 \qquad i = 1, 2, \ldots n$$

is unique, namely that the c_{lj} of system (90) is the c_l of system (91) for all j. Consequently the solution of system (89) is unique and of the type $\overline{Y}(x)C\cdot$. One solution of (89) is known, however, namely $Y^{(k)}(x)\cdot$. Inasmuch as $G^{(k)}(x, t)$ was also seen to be a solution in x it follows that each column of $G^{(k)}(x, t)$ must be the same as the general column of $Y^{(k)}(x)\cdot$ except possibly for a factor independent of x. Accordingly $g_{ij}^{(k)}(x, t) = c_j^{(k)}(t)\, y_i^{(k)}(x)$,

from which $$G^{(k)}(x, t) = \frac{1}{n}\, Y^{(k)}(x)\cdot\cdot C^{(k)}(t).$$

But from (88) it is also seen that $G^{(k)}(x, t)$ is, for fixed x, a solution in t of the system adjoint to (89). Since this solution is again unique and $\cdot Z^{(k)}(t)$ is a solution we have

$$g_{ij}^{(k)}(x, t) = y_i^{(k)}(x)\, c_j^{(k)}(t) = c_i^{(k)}(x)\, z_j^{(k)}(t),$$

i.e. $$c_j^{(k)}(t) = c^{(k)}\, z_j^{(k)}(t).$$

Hence $$g_{ij}^{(k)}(x, t) = c^{(k)}\, y_i^{(k)}(x)\, z_j^{(k)}(t)$$

or

(92) $$G^{(k)}(x, t) = \frac{c^{(k)}}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t),$$

the value of $c^{(k)}$ being as yet undetermined.

Writing equation (46) in the form

$$Y^{(k)\prime}(x)\cdot = \{R(x)\,\lambda + B(x)\}\, Y^{(k)}(x)\cdot + \{\lambda_k - \lambda\}\, R(x)\, Y^{(k)}(x)\cdot,$$

and considering this as a non-homogeneous equation, we have

$$Y^{(k)}(x)\cdot = -\{\lambda - \lambda_k\} \int_a^b G(x, t, \lambda)\, R(t)\, Y^{(k)}(t)\cdot dt.$$

Inasmuch as

$$\lim_{\lambda=\lambda_k} \{\lambda - \lambda_k\}\, G(x, t, \lambda) = G^{(k)}(x, t),$$

this yields

$$Y^{(k)}(x)\cdot = -\int_a^b G^{(k)}(x, t)\, R(t)\, Y^{(k)}(t)\cdot dt,$$

and substituting from (92) we see that

$$Y^{(k)}(x)\cdot = -\int_a^b \frac{c^{(k)}}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t)\, R(t)\, Y^{(k)}(t)\cdot dt,$$

But in view of the relation (85),

$$Y^{(k)}(x)\cdot \int_a^b \cdot Z^{(k)}(t)\, R(t)\, Y^{(k)}(t)\cdot dt =$$

$$n \left\{ \int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, \gamma_h(t)\, y_h^{(k)}(t) dt \right\} Y^{(k)}(x).$$

It follows that

$$1 = -c^{(k)} \int_a^b \sum_{h=1}^{n} z_h^{(k)}(t)\, \gamma_h(t)\, y_h^{(k)}(t)\, dt,$$

or upon associating with $Y^{(k)}(x)\cdot$ the proper $\cdot Z^{(k)}(x)$ (see page 101) that $c^{(k)} = -1$.

Hence

$$G^{(k)}(x, t) = -\frac{1}{n}\, Y^{(k)}(x)\cdot\cdot Z^{(k)}(t). \tag{93}$$

This result enables us to deduce an expression for the sum of m terms of the series development of an arbitrary vector $F(x)\cdot$. Thus we have from (77) and (78), ($\cdot Z^l$ and $Y^l\cdot$ being associated in the manner stated)

$$\int_a^b \cdot Z^{(l)}(x)\, R(x)\, F(x)\cdot dx = c_l(1).$$

Multiplying this equation by $Y^{(l)}(x)\cdot$ on the left it becomes

$$c_l Y^{(l)}(x)\cdot(1) = \int_a^b Y^{(l)}(x)\cdot\cdot Z^{(l)}(t)\, R(t)\, F(t)\cdot dt,$$

which in view of (85) and (89) reduces to

$$(94)\qquad c_l Y^{(l)}(x)\cdot\; = -\int_a^b G^{(l)}(x, t)\, R(t)\, F(t)\cdot dt.$$

This expression for the l^{th} term of the formal development was deduced upon the hypothesis that λ_l is a simple characteristic value. If at λ_l a number of characteristic values $\lambda_{l_1}, \lambda_{l_2}, \ldots \lambda_{l_\mu}$, coincide, we shall define the term of the formal series which corresponds to this value of λ to be

$$-\int_a^b G^{(l)}(x, t)\, R(t)\, F(t)\cdot dt.$$

In every case then we have a formal series which is completely determined and of which the l^{th} term is given by formula (94). It is to be noted that $G^{(l)}(x, t)$ for the case in which λ_l is not a simple characteristic value is not given by the left-hand side of (94). It must be computed for every such case individually.

Now let ρ_l be any contour in the λ plane which surrounds λ_l but no other characteristic values. Then by use of the relation

$$\frac{1}{2\pi i}\int_{\rho_l} G(x, t, \lambda)\, d\lambda = G^{(l)}(x, t)$$

we have from (94)

$$c_l\, Y^{(l)}(x)\cdot\; = -\frac{1}{2\pi i}\int_a^b \int_{\rho_l} G(x, t, \lambda)\, d\lambda\, R(t)\, F(t)\cdot dt,$$

or, upon interchanging the order of integration and choosing a contour C_m which encloses the characteristic values $\lambda_1, \lambda_2, \ldots \lambda_m$ and no others,

$$(95)\qquad \sum_{k=1}^{m} c_k\, Y^{(k)}(x)\cdot\; = -\frac{1}{2\pi i}\int_{C_m}\int_a^b G(x, t, \lambda)\, R(t)\, F(t)\cdot dt\, d\lambda.$$

SECTION IX.

The Convergence of the expansion.

From the distribution of characteristic values as found in section VII, for both the cases of regular and irregular conditions there discussed, it becomes readily apparent that given any constant N sufficiently large it is always possible to choose a circle C whose center is at $\lambda = 0$, and whose radius both exceeds the value N and is such that the distance from any point of C to a characteristic value is greater than some fixed constant $\bar{\delta} > 0$.

We shall consider the convergence of the contour integral in formula (95) as the contour of integration is taken successively as a larger and larger circle of the type C above. With the size of the circle the number of characteristic values which it includes and hence the number of terms of the series which are summed by the integration may be increased indefinitely, the limit of the integral for $|\lambda| = \infty$, being, if it exists, the sum of the corresponding series.

Let us recall the hypotheses already made concerning the functions $\gamma_i(x)$. We have

$$(96)\begin{cases} \text{(i)} & \gamma_i(x) \neq 0 \\ \text{(ii)} & \gamma_i(x) \text{ continuous } a \leqq x \leqq b \\ \text{(iii)} & \text{if n} \geqq 2 \quad \gamma_i(x) \neq \gamma_j(x) \text{ for } i \neq j \\ \text{(iv)} & \text{if } n \geqq 2 \quad \arg\{\gamma_i(x) - \gamma_j(x)\} = h_{ij} \end{cases} \qquad \begin{matrix} \text{(page 72)} \\ \\ \text{(page 83)} \end{matrix}$$

To these we shall add

$$(97) \qquad \arg \gamma_i(x) = H_i \text{ (a constant)}, \quad i = 1, 2, \ldots n.$$

Concerning the vector to be expanded we shall assume

(98) that the elements $f_i(x)$, $i = 1, 2, \ldots n$, consist in the interval $a \leqq x \leqq b$, of only a finite number of pieces each of which is real, continuous, and has a continuous derivative.

It will readily be seen from (96) and (97) that we have restricted each $\gamma_j(x)$ to vary along a ray in the complex plane. Moreover, if $n \geqq 2$ the dependence of $\gamma_j(x)$ upon x must be such that the slope of the line joining any two of these points remains constant. This

means that every n sided polygon with vertices at the points $\gamma_j(x_0)$, $j = 1, 2, \ldots n$, $a \leqq x_0 \leqq b$, at most expands or contracts about $\lambda = 0$ as x_0 is allowed to vary.

If the lines along which $\gamma_j(x)$, $j = 1, 2, \ldots n$, vary are all distinct, the conditions of the system are regular provided W_a and W_b are suitably chosen. If, however, one or more sides of any of the polygons mentioned lie on a line through the point $\lambda = 0$, we have the irregular case discussed in section VII. In particular all the sides may lie on such a line, (for instance the quantities $\gamma_j(x)$ may be all real) and it is upon this configuration that the irregular case in question bases its chief claim for interest. Since the condition (96 iv) is automatically fulfilled in this case the functional dependence of $\gamma_j(x)$ upon x is far less restricted than when the points $\gamma_j(x)$ form the vertices of actual polygons.

Substituting in formula (86) the value $\bar{G}(x, t) = \pm Y(x)\, Z(t)$ we have

$$G(x, t, \lambda) = Y(x) \left\{ \pm \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1}\{W_a Y(a) - W_b Y(b)\}\right\} Z(t),$$

the upper sign holding for $t < x$ and the lower sign for $t > x$. It is desirable in the following work to express $G(x, t, \lambda)$ in a somewhat different form.

Upon setting

$$+ \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1} \{ W_a Y(a) - W_b Y(b)\} = (\delta^*_{ij}) + U,$$

multiplication by Δ yields

$$\tfrac{1}{2} W_a Y(a) + \tfrac{1}{2} W_b Y(b) + \tfrac{1}{2} W_a Y(a) - \tfrac{1}{2} W_b Y(b) = \\ \{W_a Y(a) + W_b Y(b)\}\, (\delta^*_{ij}) + \Delta\, U,$$

which, in view of the relation $(\delta^*_{ij}) + (\delta^{**}_{ij}) = I$, reduces to

$$W_a Y(a)\, (\delta^{**}_{ij}) - W_b Y(b)\, (\delta^*_{ij}) = \Delta\, U.$$

Hence

$$U = \Delta^{-1}\, W_a Y(a)\, (\delta^{**}_{ij}) - \Delta^{-1}\, W_b Y(b)\, (\delta^*_{ij}).$$

Now setting

$$- \tfrac{1}{2} I + \tfrac{1}{2} \Delta^{-1}\{W_a Y(a) - W_b Y(b)\} = - (\delta^{**}_{ji}) + V,$$

it is found by precisely the same method that $V = U$. In consequence we have for the Green's function

$$(99) \quad G(x, t, \lambda) = Y(x) \left\{ \left\{ \begin{matrix} (\delta^*_{ji}) \\ \text{or} \\ -(\delta^{**}_{ij}) \end{matrix} \right\} + \Delta^{-1} W_a Y(a)\,(\delta^{**}_{ij}) - \Delta^{-1} W_b Y(b)(\delta^*_{ij}) \right\} Z(t),$$

where the upper form is to be chosen when $t < x$, and the lower one for $t > x$.

By means of this formula $G(x, t, \lambda)$ may be explicitly represented by choosing as the solutions $Y(x)$ and $Z(t)$ a pair which are analytic in λ and have the forms

$$Y(x) = \left(e^{\lambda \Gamma_j(x) + B_j(x)} \left\{ \delta_{ij} + \frac{[\varphi_{ij}(x)]}{\lambda} \right\} \right)$$
$$Z(t) = \left(e^{-\lambda \Gamma_i(t) - B_i(t)} \left\{ \delta_{ij} + \frac{[\psi_{ij}(t)]}{\lambda} \right\} \right),$$

obtained in sections V and VI. It should be observed that $G(x, t, \lambda)$ is unique (see page 70), despite the fact that if $n \geqq 2$ the choice of $Y(x)$ and $Z(t)$ thus determined upon changes from any one to any other of the sectors within which no quantities $R\{\lambda\{\gamma_i(x) - \gamma_j(x)\}\}$ change sign, and despite the fact that the values of δ^*_{ij} and δ^{**}_{ij} change from any one to any other sector $\sigma_{\mu\nu}$.

From (95) we have, upon denoting by $S_m(x)\cdot$ the sum of those terms of the formal development of $F(x)\cdot$ which correspond to the characteristic values enclosed by the circle C in question,

$$(100) \qquad S_m(x)\cdot = \frac{-1}{2\pi i} \int_C \int_a^b G(x, t, \lambda)\, R(t)\, F(t) \cdot dt\, d\lambda.$$

Substituting in this the value of $G(x, t, \lambda)$ as found above in (99) it is seen that $S_m(x)\cdot = \sum_{i=1}^{4} S^{(i)}_m(x)\cdot$, where the quantities $S^{(i)}_m(x)\cdot$ are given by the relations

$$
(101)\quad \begin{cases}
S_m^{(1)}(x)\cdot = \sum\limits_C \dfrac{-1}{2\pi i}\displaystyle\int\limits_{C_{\mu\nu}}\int\limits_a^x Y(x)(\delta_{ij}^*)Z(t)R(t)F(t)\cdot dt\, d\lambda \\[2ex]
S_m^{(2)}(x)\cdot = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int\limits_{C_{\mu\nu}}\int\limits_x^b Y(x)(\delta_{ij}^{**})Z(t)R(t)F(t)\cdot dt\, d\lambda \\[2ex]
S_m^{(3)}(x)\cdot = \sum\limits_C \dfrac{-1}{2\pi i}\displaystyle\int\limits_{C_{\mu\nu}} Y(x)\Delta^{-1}\int\limits_a^b W_a Y(a)(\delta_{ij}^{**})Z(t)R(t)F(t)\cdot dt d\lambda \\[2ex]
S_{mt}^{(4)}(x)\cdot = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int\limits_{C_{\mu\nu}} Y(x)\Delta^{-1}\int\limits_a^b W_b Y(b)\,(\delta_{ij}^*)Z(t)\,R(t)F(t)\cdot dt d\lambda,
\end{cases}
$$

$C_{\mu\nu}$ denoting any arc of circle C which lies within the sector $\sigma_{\mu\nu}$ and, if $n \geqq 2$, upon which none of the quantities $R\,\{\lambda\{\gamma_i(x) - \gamma_j(x)\}$ change sign. If $C_{\mu\nu}$ abuts upon a ray bounding $\sigma_{\mu\nu}$ it shall either include or exclude the end point for which the quantity $R\,\{\lambda\Gamma_\mu(b)\}$ or the quantity $R\,\{\lambda\Gamma_\nu(b)\}$ vanishes according as the quantity in question is < 0 or > 0 within $\sigma_{\nu\mu}$. The arc $C_{\mu\nu}$ may, therefore, include one, both, or neither end point, and since the reasoning is precisely the same in each case (the sector can be split if both end points are included) we shall consider for the sake of concreteness that it includes one, namely that one which lies on the ray $R\,\{\lambda\Gamma_\nu(b)\} = 0$. The symbol $\sum\limits_C$ indicates that the sum of the integrals over all arcs composing the circle C is to be taken.

We shall proceed to evaluate each of the integrals above in turn, and in the course of this evaluation it will be convenient to refer to the facts established by the following lemmas. The notation $|\,\varphi(x)\,| < M$ will be used to indicate that the function in question is bounded. It is not to be understood that the M in any one case represents the same constant as in any other, but merely that there exists in each case some constant for which the relation is true.

Lemma 1: Given any function $\varphi_1(x, \lambda)$ which is such that

$$
(102)\quad \begin{cases}
\text{(i)} & |\,\varphi_1(x,\lambda)\,| < M \text{ for } \alpha \leqq x \leqq \beta,\ |\,\lambda\,| > N, \\
\text{(ii)} & \lim\limits_{\lambda=\infty} |\,\varphi_1(x,\lambda)\,| = 0 \text{ uniformly, for } \alpha \leqq x \leqq \beta - \chi,
\end{cases}
$$

where χ is an arbitrarily small positive constant, then

$$\lim_{|\lambda|=\infty} \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx = 0\,.$$

Proof: Under the hypotheses (102) we have

$$\lim_{|\lambda|=\infty} \int_{\alpha}^{\beta-\chi} \varphi_1(x, \lambda)\, dx = 0 \text{ uniformly,}$$

and

$$\left| \int_{\beta-\chi}^{\beta} \varphi_1(x, \lambda)\, dx \right| < M\chi,$$

while it follows from these relations and the relation

$$\left| \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx \right| \leqq \left| \int_{\alpha}^{\beta-\chi} \varphi_1(x, \lambda)\, dx \right| + \left| \int_{\beta-\chi}^{\beta} \varphi_1(x, \lambda)\, dx \right|,$$

that

$$\left| \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx \right| < 2M\chi$$

for $|\lambda|$ sufficiently large. Since χ is arbitrary, however, this means that

$$\lim_{|\lambda|=\infty|} \int_{\alpha}^{\beta} \varphi_1(x, \lambda)\, dx = 0. \qquad \text{Q. E. D.}$$

Lemma 2: Given any function $\varphi_2(\lambda)$ which is such that

$$(103)\begin{cases} \text{(i)} & |\varphi_2(\lambda)| < M \text{ for } \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta,\ |\lambda| > N \\ \text{(ii)} & \lim\limits_{|\lambda|=\infty} |\varphi_2(\lambda)| = 0 \text{ uniformly, for } \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta - \theta_\chi, \end{cases}$$

where θ_χ is a constant arbitrarily small but positive, then

$$\lim_{|\lambda|=\infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda) \frac{d\lambda}{\lambda} = 0,$$

$C_{\alpha\beta}$ being that arc of the circle $|\lambda| = \rho$ which lies in the sector bounded by $\arg\lambda = \theta_\alpha$ and $\arg\lambda = \theta_\beta$.

Proof: Writing $\lambda = \rho e^{i\theta}$ we have

$$\varphi_2(\lambda) = \varphi_2(\rho e^{i\theta}) = \psi_2(\theta, \rho).$$

Hence

$$\lim_{|\lambda|=\infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda) \frac{d\lambda}{\lambda} = \lim_{\rho=\infty} \frac{1}{2\pi} \int_{\theta_\alpha}^{\theta_\beta} \psi_2(\theta, \rho)\, d\theta,$$

and since the limit of the integral on the right is zero by lemma 1,

$$\lim_{|\lambda|=\infty} \frac{1}{2\pi i} \int_{C_{\alpha\beta}} \varphi_2(\lambda) \frac{d\lambda}{\lambda} = 0. \qquad \text{Q. E. D.}$$

Lemma 3: Given any function $\varphi_3(x, \lambda)$ which is such that

$$(104)\quad \begin{cases} \text{(i)} \quad |\varphi_3(x,\lambda)| < M \text{ for } \alpha \leqq x \leqq \beta, \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta, |\lambda| > N, \\ \text{(ii)} \quad \lim\limits_{|\lambda|=\infty} \varphi_3(x,\lambda)| = 0 \text{ uniformly for } \alpha \leqq x \leqq \beta - \chi, \\ \qquad\qquad \theta_\alpha \leqq \arg\lambda \leqq \theta_\beta - \theta_\chi, \end{cases}$$

then $I(\lambda) = \int_\alpha^\beta \varphi_3(x, \lambda)\, dx$ is a function of the type $\varphi_2(\lambda)$ of lemma 2.

Proof: When λ is confined to the sector $\theta_\alpha \leqq \arg\lambda \leqq \theta_\beta - \theta_\chi$, $\lim\limits_{|\lambda|=\infty} I(\lambda) = 0$ uniformly by lemma 1, while for λ in the larger sector $\theta_\alpha \leqq \arg\lambda \leqq \theta_\beta$

$$|I(\lambda)| \leqq \int_\alpha^\beta M dx = M(\beta - \alpha).$$

Hence $I(\lambda)$ is of type $\varphi_2(\lambda)$ by definition. Q. E. D.

As a matter of convenience we shall use hereafter the symbols φ_1, φ_2, and φ_3, to designate any function of λ and other arguments which satisfies conditions similar to (102), (103) or (104) respectively. As heretofore functions which approach the limit zero uniformly as $|\lambda|$ increases indefinitely will be denoted by ϵ. Let us return now to the direct evaluation of $S_m(x)$·.

I. $S_m^{(1)}(x)\cdot$.

Writing

$$J_1\cdot = -\int_a^x Y(x)\,(\delta_{ij}^*)\,Z(t)\,R(t)\,F(t)\cdot dt$$

we have from (98)

$$S_m^{(1)}(x)\cdot = \frac{1}{2\pi i}\int_C J_1\cdot d\lambda.$$

From this it is immediately seen that

$$(105)\qquad S_m^{(1)}(a)\cdot = O.$$

Consequently it is necessary to consider further only the case $x \mp a$. We have

$$J_1\cdot = -\left(\int_a^x \sum_{k,m,p,q=1}^{n} e^{\lambda\Gamma_k(x)+B_k(x)}\{\delta_{ik}+1/\lambda[\varphi_{ik}(x)]\}\delta_{km}^* e^{-\lambda\Gamma_m(t)-B_m(t)} \{\delta_{mp}+1/\lambda[\psi_{mp}(t)]\}\delta_{pq}\gamma_q(t)\,f_q(t)\,dt\right),$$

i.e.

$$(106)\quad J_1\cdot = -\left(\int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}+B_i(x)-B_i(t)}\,\delta_{ii}^*\gamma_i(t)\,f_i(t)\,dt\right) + 1/\lambda\left(\int_a^x \sum_{k,p=1}^{n} e^{\lambda\{\Gamma_k(x)-\Gamma_k(t)\}+B_k(x)-B_k(t)}\,\delta_{kk}^*\,\gamma_p(t)\,f_p(t)[\delta_{kp}\,\varphi_{ik}(x) + \delta_{ik}\,\psi_{kp}(t)]\,dt\right).$$

For λ on any arc $C_{\mu\nu}$, however, each term under the sign of summation in the last matrix on the right of this expression is seen to be of the type $\varphi_3(t,\lambda)$. Their sum therefore is of the same type, and by lemma 2 the integral is of type $\varphi_2(\lambda)$. Consequently we have, upon integrating by parts the elements in the first matrix on the right of (106),[15]

$$J_1\cdot = \frac{1}{\lambda}\left(-\delta_{ii}^*\left\{-f_i(x-0)+e^{\lambda\Gamma_i(x)+B_i(x)}f_i(a+0) + \int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)}\frac{d}{dt}\left\{f_i(t)e^{B_i(x)-B_i(t)}\right\}dt\right\}\right)+\frac{(\varphi_2)}{\lambda}.$$

15 Recall that $\Gamma_i(a) = B_i(a) = 0$.

Accordingly it is seen that $J_1\cdot$ can be considered as a sum of matrices of which the first three are discussed as follows.

For the first matrix we have directly

$$\frac{1}{\lambda}\,(\overset{*}{\delta}_{ii} f_i(x-0)) = \frac{1}{\lambda}\,(\overset{*}{\delta}_{ij})\, F(x-0)\cdot.$$

The second matrix

$$-\frac{1}{\lambda}\,(\overset{*}{\delta}_{ii}\, e^{\lambda\Gamma_i(x)+B_i(x)}\, f_i(a))$$

is one each of whose elements is seen either to vanish or to be of type $\varphi_2(\lambda)/\lambda$ on arc $C_{\mu\nu}$. Hence the entire matrix may be represented by $(\varphi_2)/\lambda$.

The third matrix

$$-\frac{1}{\lambda}\left(\overset{*}{\delta}_{ii}\int_a^x e^{\lambda\{\Gamma_i(x)-\Gamma_i(t)\}}\,\frac{d}{dt}\{f_i(t)\,e^{B_i(x)-B_i(t)}\}\,dt\right)$$

is likewise one each of whose elements either vanishes or is, on arc $C_{\mu\nu}$, of type φ_2/λ by lemma 3. This matrix is, therefore, also of the type $(\varphi_2)/\lambda$, and inasmuch as a sum of matrices of type (φ_2) is again a matrix (φ_2) we have

$$J_1\cdot = \frac{1}{\lambda}\,\{(\overset{*}{\delta}_{ij})\,F(x-0)\cdot + (\varphi_2)\}$$

The integration of $J_1\cdot$ over the arc $C_{\mu\nu}$ yields, therefore, by lemma 2

$$\frac{1}{2\pi i}\int_{C_{\mu\nu}}\left\{(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\varphi_2)\right\}\frac{d\lambda}{\lambda} = \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\epsilon),$$

where $\omega_{\mu\nu}$ is the angle subtended by $C_{\mu\nu}$ at $\lambda = 0$.

A similar expression results from integration over each arc similar in character to $C_{\mu\nu}$. Hence we have

$$S_m^{(1)}(x)\cdot = \sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(x-0)\cdot + (\epsilon).$$

Let us consider in detail the sum

$$\sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)}). \tag{107}$$

Corresponding to each arc $C_{\mu\nu}$ there exists another arc $C_{\bar{\mu}\bar{\nu}}$ which is the reflection of $C_{\mu\nu}$ in the point $\lambda = 0$. Since $C_{\bar{\mu}\bar{\nu}}$ also subtends the angle $\omega_{\mu\nu}$ at the point $\lambda = 0$ we can write the sum (107) equally well in the form

$$(108) \qquad \sum_{\frac{1}{2}C} \frac{\omega_{\mu\nu}}{2\pi} (\overset{*}{\delta}{}_{ij}^{(\mu\nu)} + \overset{*}{\delta}{}_{ij}^{(\bar{\mu}\bar{\nu})}),$$

the summation covering now the arc of only one (any) half of circle C. But it will readily be seen that $\overset{*}{\delta}{}_{ij}^{(\bar{\mu}\bar{\nu})} = \overset{**}{\delta}{}_{ij}^{(\mu\nu)}$, for quantities which have a positive real part in $\sigma_{\mu\nu}$ have naturally a negative real part in $\sigma_{\bar{\mu}\bar{\nu}}$. Hence $\overset{*}{\delta}{}_{ij}^{(\mu\nu)} + \overset{*}{\delta}{}_{ij}^{(\bar{\mu}\bar{\nu})} = \delta_{ij}$, and the sum (108) i.e. (107) reduces to

$$\sum_{\frac{1}{2}C} \frac{\omega_{\mu\nu}}{2\pi} I = \tfrac{1}{2} I.$$

It follows, therefore, that

$$(109) \qquad \sum_{C} \frac{\omega_{\mu\nu}}{2\pi} (\overset{*}{\delta}{}_{ij}^{(\mu\nu)}) = \tfrac{1}{2} I.$$

In consequence we have

$$S_m^{(1)}(x)\cdot = \tfrac{1}{2} F(x-0)\cdot + (\epsilon),$$

i.e.

$$(110) \qquad \lim_{m=\infty} S_m^{(1)}(x)\cdot = \tfrac{1}{2} F(x-0)\cdot$$

II. $S_m^{(2)}(x)\cdot$.

The treatment of this expression is parallel to that of $S_m^{(1)}(x)\cdot$ and is as follows.

Writing

$$J_2\cdot = \int_x^b Y(x)\,(\overset{**}{\delta}_{ij}) Z\,(t)\,R(t)\,F(t)\cdot dt,$$

we have from (101) $\qquad S_m^{(2)}(x)\cdot = \dfrac{1}{2\pi i}\displaystyle\int_C J_2\cdot d\lambda.$

Hence

$$(111) \qquad S_m^{(2)}(b)\cdot = 0,$$

and it is necessary to consider further only the case $x \neq b$. We have

$$J_2 \cdot = \left(\int_x^b \sum_{k,m,p,q=1}^{n} e^{\lambda \Gamma_k(x) + B_k(x)} \left\{ \delta_{ik} + \frac{[\varphi_{ik}(x)]}{\lambda} \right\} \delta_{km}^{**} e^{-\lambda \Gamma_m(t) - B_m(t)} \left\{ \delta_{mp} + \frac{[\psi_m(t)]}{\lambda} \right\} \delta_{pq} \gamma_q(t) f_q(t)\, dt \right)$$

$$= \left(\int_x^b e^{\lambda \{ \Gamma_i(x) - \Gamma_i(t) \} + B_i(x) - B_i(t)} \delta_{ii}^{**} \gamma_i(t) f_i(t)\, dt \right) +$$

$$\frac{1}{\lambda} \left(\int_x^b \sum_{k,p=1}^{n} e^{\lambda \{ \Gamma_k(x) - \Gamma_k(t) \} + B_k(x) - B_k(t)} \delta_{kk}^{**} \gamma_p(t) f_p(t)\, [\delta_{kp} \varphi_{ik}(x) + \delta_{ik} \psi_{kp}(t)]\, dt \right).$$

Each integrand in the last matrix of this equation is of the type $\varphi_1(t, \lambda)$ and accordingly the matrix is, by lemma 1 of type $(\epsilon)/\lambda$. Integrating by parts the elements of the first matrix on the right we have

$$J_2 \cdot = \frac{1}{\lambda} \left(\delta_{ii}^{**} \left\{ - e^{\lambda \{ \Gamma_i(x) - \Gamma_i(b) \} + B_i(x) - B_i(b)} f_i(b - 0) + f_i(x + 0) + \int_x^b e^{\lambda \{ \Gamma_i(x) - \Gamma_i(t) \}} \frac{d}{dt} \{ f_i(t)\, e^{B_i(x) - B_i(t)} \}\, dt \right\} \right) + \frac{(\epsilon)}{\lambda}.$$

Therefore, $J_2 \cdot$ can also be expressed as a sum of matrices, the first and third of which, namely

$$\frac{1}{\lambda} \left(- \delta_{ii}^{**} e^{\lambda \{ \Gamma_i(x) - \Gamma_i(b) \} + B_i(x) - B_i(b)} f_i(b) \right)$$

and

$$\frac{1}{\lambda} \left(\delta_{ii}^{**} \int_x^b e^{\lambda \{ \Gamma_i(x) - \Gamma_i(t) \}} \frac{d}{dt} \{ f_i(t)\, e^{B_i(x) - B_i(t)} \}\, dt \right),$$

are of the type $(\epsilon)/\lambda$ on arc $C_{\mu\nu}$, for each element either vanishes or approaches the limit zero as $|\lambda| = \infty$. The second matrix of the sum is directly

$$\frac{1}{\lambda}\,(\overset{**}{\delta_{ii}}\; f_i(x+0)) = \frac{1}{\lambda}\,(\overset{**}{\delta_{ij}})\; F(x+0)\cdot.$$

Hence

$$J_2\cdot = \frac{1}{\lambda}\,\{(\overset{**}{\delta_{ij}})\; F(x+0)\cdot + (\epsilon)\},$$

which integrated over $C_{\mu\nu}$ gives

$$\frac{1}{2\pi i}\int\limits_{C_{\mu\nu}}\left\{(\overset{**}{\delta_{ij}^{(\mu\nu)}})\; F(x+0)\cdot + (\epsilon)\right\}\frac{d\lambda}{\lambda} = \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta_{ij}^{(\mu\nu)}})\; F(x+0)\cdot + (\epsilon).$$

By familiar reasoning this leads to the equation

$$S_m^{(2)}(x)\cdot = \sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta_{ij}^{(\mu\nu)}})\; F(x+0)\cdot + (\epsilon),$$

and inasmuch as

(112) $$\sum_C \frac{\omega_{\mu\nu}}{2\pi}\,(\overset{**}{\delta_{ij}^{(\mu\nu)}}) = \tfrac{1}{2}\, I,$$

as may be shown by applying again the argument by means of which relation (109) was established, we have

$$S_m^{(2)}(x)\cdot = \tfrac{1}{2}\, F(x+0)\cdot + (\epsilon),$$

i.e.

(113) $$\lim_{m\to\infty} S_m^{(2)}(x)\cdot = \tfrac{1}{2}\, F(x+0)\cdot$$

III. $S_m^{(3)}(x)\cdot$.

Following the procedure used in the discussion of the preceding expressions, let $J_3\cdot$ be defined by the equation

$$J_3\cdot = -\,Y(x)\,\Delta^{-1}\int_a^b W_a Y(a)\,(\overset{**}{\delta_{ij}})\; Z(t)\; R(t)\bar{F}(t)\cdot dt.$$

Then $$S_m^{(3)}(x)\cdot = \frac{1}{2\pi i}\int_C J_3\cdot d\lambda.$$

Now

$$\Delta^{-1} = \left(\frac{D_{ji}(\lambda)}{D(\lambda)}\right),$$ [16]

[16] See note on page 104.

and substituting for $D(\lambda)$ its value as given by equation (56) we have

$$\Delta^{-1} = \left\| \frac{\prod_{\substack{k=1 \\ k \neq i}}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\overset{**}{\delta}{}_{kk}^{(\mu\nu)}} \bar{D}_{ji}(\lambda)}{\prod_{k=1}^{n} e^{\{\lambda\Gamma_k(b)+B_k(b)\}\overset{**}{\delta}{}_{kk}^{(\mu\nu)}} \bar{D}(\lambda)} \right\|$$

$$= \left\| e^{-\{\lambda\Gamma_i(b)+B_i(b)\}\overset{**}{\delta}{}_{ii}^{(\mu\nu)}} \frac{\bar{D}_{ji}(\lambda)}{\bar{D}(\lambda)} \right\|,$$

where $\bar{D}(\lambda)$ is given for the case of regular conditions by formula (57) and for the case of irregular conditions discussed in section VII by either formula (66) or formula (68). A glance at these formulas shows, however, that in each case the expression for $\bar{D}(\lambda)$ reduces, under the conditions for which the expression is valid to the form

$$\bar{D}(\lambda) = W^{(\mu\nu)} + \varphi_2, \text{ on arc } C_{\mu\nu}.$$

Now every point of a circle C, and hence in particular of an arc $C_{\mu\nu}$ is at a distance which exceeds a fixed quantity from any characteristic value. However, by analogy with formula (59), for any point of $C_{\mu\nu}$ under regular conditions for which the function $\bar{D}$ is sufficiently small we have

$$\lambda = \frac{1}{\Gamma_\nu(b)} \left\{ -B_\nu(b) + \log \frac{-W^{(\mu\nu)}}{W^{(\nu\tau)}} + \epsilon + 2p\pi i \right\}.$$

In other words the point in question will necessarily lie near one of the characteristic values.

This stands in contradiction with the fundamental property of the circle C.

Inasmuch as a similar relation holds on every arc of type $C_{\mu\nu}$, it follows that for every point of all circles C, $\bar{D}(\lambda)$ exceeds, under regular conditions, some fixed positive constant δ_1, i.e.

(114) $$| \bar{D}(\lambda) | \geqq \delta_1 > 0 \quad \text{for } \lambda \text{ on any circle } C.$$

While we have thus proved the relation (114) only for the case of regular conditions it can be shown to hold equally well under the type of

irregular conditions considered in section VII.[17] In either case, therefore, Δ^{-1} is uniformly bounded on all circles of type C.

Proceeding, we have

$$Y(x)\Delta^{-1} = \left\| \sum_{k=1}^{n} e^{\lambda\Gamma_k(x)+B_k(x)} \left\{ \delta_{ik} + \frac{[\varphi_{ik}(x)]}{\lambda} \right\} e^{-\{\lambda\Gamma_k(b)+B_k(b)\} \overset{**}{\delta}_{kk}} \frac{\overline{D}_{jk}(\lambda)}{\overline{D}(\lambda)} \right\|$$

$$= \left\| e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \left\{ \frac{\overline{D}_{ji}(\lambda)}{\overline{D}(\lambda)} \right\} \right\|$$

$$+ \left\| \sum_{k=1}^{n} e^{\lambda\{\Gamma_k(x)-\Gamma_k(b)\overset{**}{\delta}_{kk}\}+B_k(x)-B_k(b)\overset{**}{\delta}_{kk}} \frac{[\varphi_{ik}(x)]}{\lambda} \frac{\overline{D}_{jk}(\lambda)}{D(\lambda)} \right\|.$$

Since the last matrix of this expression is of type (ϵ) we have

$$Y(x)\Delta^{-1} = \left(e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \left\{ \frac{\overline{D}_{ji}(\lambda)}{D(\lambda)} + \epsilon \right\} \right),$$

or, more briefly,

$$Y(x)\Delta^{-1} = \left(e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \tau_{ij}(\lambda)\right), \tag{115}$$

where $\qquad | \tau_{ij}(\lambda) | < M \quad \text{for} \quad | \lambda | > N.$

Inasmuch as $\overline{D}(\lambda)$ was seen to be of the form

$$\overline{D}(\lambda) = W^{(\mu\nu)} + \varphi_2$$

for λ on arc $C_{\mu\nu}$, we have for any λ on this arc

$$(\tau_{ij}) = \left(\frac{W_{ji}^{(\mu\nu)} + \varphi_2}{W^{(\mu\nu)} + \varphi_2} + \epsilon \right) = \left(\frac{W_{ji}^{(\mu\nu)}}{W^{(\mu\nu)}} \right) + (\varphi_2),$$

or, denoting by $\Omega^{(\mu\nu)}$ the matrix whose determinant is $W^{(\mu\nu)}$,

$$(\tau_{ij}) = \overset{-1}{\Omega^{(\mu\nu)}} + (\varphi_2). \tag{116}$$

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[17] Cf. Wilder, loc. cit. p. 422.

We have, further,

$$\int_a^b W_a Y(a)\,(\delta_{ij}^{**})\,Z(t)\,R(t)\,F(t)\cdot dt =$$

$$\left[\int_a^b \sum_{h,k,m,p,q=1}^{n} w_{ih}^{(a)} \left\{ \delta_{hk} + \frac{[\varphi_{hk}(a)]}{\lambda} \right\} \delta_{km}^{**} e^{-\lambda\Gamma_m(t)-B_m(t)} \left\{ \delta_{mp} + \frac{[\psi_{mp}(t)]}{\lambda} \right\} \delta_{pq}\,\gamma_q(t)\,f_q(t)\,dt) \right] =$$

$$\left[\sum_{h=1}^{n} w_{ih}^{(a)}\delta_{hh}^{**} \int_a^b e^{-\lambda\Gamma_h(t)-B_h(t)}\,\gamma_h(t)\,f_h(t)\,dt)\right] +$$

$$\frac{1}{\lambda}\left[\sum_{h,k,p=1}^{n} w_{ih}^{(a)}\delta_{kk}^{**} \int_a^b e^{-\lambda\Gamma_k(t)-B_k(t)}\,\gamma_p(t)\,f_p(t)\,[\delta_{hk}\psi_{kp}(t) + \delta_{kp}\,\varphi_{hk}^{(a)}]\,dt\right]$$

$$= \frac{1}{\lambda}\left[\sum_{h=1}^{n} w_{ih}^{(a)}\delta_{hh}^{**} \left\{ -e^{-\lambda\Gamma_h(b)-B_h(b)} f_h(b-0) + f_h(a+0) + \int_a^b e^{-\lambda\Gamma_h(t)} \frac{d}{dt}\left\{ f_h(t)\,e^{-B_h(t)} \right\} dt \right\}\right] + \frac{(\epsilon)}{\lambda},$$

the last form representing as heretofore the result of an integration by parts. The multiplication of this matrix by the matrix (115) yields the result

$$J_3 \cdot = \frac{-1}{\lambda}\left[\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\delta_{ii}^{**}\}+B_i(x)-B_i(b)\delta_{ii}^{**}}\tau_{ik}(\lambda)\,w_{kh}^{(a)}\delta_{hh}^{**} \cdot \left\{ f_h(a+0) - e^{-\lambda\Gamma_h(b)-B_h(b)} f_h(b-0) + \int_a^b e^{-\lambda\Gamma_h(t)} \frac{d}{dt}\{f_h(t)e^{-B_h(t)}\}dt \right\}\right] + \frac{(\epsilon)}{\lambda},$$

the principal matrix on the right again breaking up into three component matrices. Of these the first,

$$\frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\tau_{ik}(\lambda)w_{kh}^{(a)}\delta_{hh}^{**}e^{\lambda\Gamma_h(b)-B_h(b)}f_h(b-0)\right),$$

and the third,

$$\frac{-1}{\lambda}\left[\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}} \tau_{ik}(\lambda)\, w_{kh}^{(a)}\delta_{hh}^{**}\int_a^b e^{-\lambda\Gamma_h(t)}\frac{d}{dt}\{e^{-B_h(t)}f_h(t)\}dt\right],$$

are for all values of x clearly of the type $(\epsilon)/\lambda$. Only the second matrix of the sum, namely

$$\bar{J}_3(x)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta}_{ii}\}+B_i(x)-B_i(b)\overset{**}{\delta}_{ii}}\tau_{ik}(\lambda)w_{kh}^{(a)}\delta_{hh}^{**}f_h(a+0)\right),$$

remains, therefore, to be considered. It depends for its character upon the value of x.

For $x \neq a$, $x \neq b$, the matrix is clearly of type $(\varphi_2)/\lambda$. For $x = a$, on the other hand, the exponential factor common to the elements of the i^{th} row reduces to $e^{-\{\lambda\Gamma_i(b)+B_i(b)\}\overset{**}{\delta}_{ii}} = \delta_{ii}^{*} + \epsilon$, so that

$$\bar{J}_3(a)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n}\delta_{ii}^{*}\,\tau_{ik}(\lambda)\,w_{kh}^{(a)}\,\delta_{hh}^{**}f_h(a+0)+\epsilon\right).$$

The fact that

$$\bar{J}_3(b)\cdot=\frac{-1}{\lambda}\left(\sum_{k,h=1}^{n}\delta_{ii}^{**}\tau_{ik}(\lambda)\,w_{kh}^{(a)}\,\delta_{hh}^{**}\,f_h(a+0)+\varphi_2\right),$$

may be established in similar manner by use of the relation

$$e^{\{\lambda\Gamma_i(b)+B_i(b)\}\delta_{ii}^{*}} = \delta_{ii}^{**}+\varphi_2.$$

Substituting for (τ_{ij}) its value as given by formula (115) it is readily seen, therefore, that on arc $C_{\mu\nu}$

$$\bar{J}_3(a)\cdot = \frac{-1}{\lambda}(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,F(a+0)\cdot + (\varphi_2)\}$$

$$\bar{J}_3(b)\cdot = \frac{-1}{\lambda}\{(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,F(a+0)\cdot + (\varphi_2)\}.$$

It follows from this that

$$S_m^{(3)}(x)\cdot\begin{cases} = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}(\epsilon)\frac{d\lambda}{\lambda} & \text{when } x \neq a,\ x \neq b. \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}\{-(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,F(a+0)\cdot + (\varphi_2)\}\frac{d\lambda}{\lambda} & \\ & \text{when } x = a \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}}\{-(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,F(a+0)\cdot + (\varphi_2)\}\frac{d\lambda}{\lambda} & \\ & \text{when } x = b, \end{cases}$$

or, upon defining the matrices $\overline{K}_3$ and $\overline{\overline{K}}_3$ by the equations

$$\text{(117)}\qquad \begin{cases} \overline{K}_3 = \sum\limits_C \left\{-\dfrac{w_{\mu\nu}}{2\pi}(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\right\} \\ \overline{\overline{K}}_3 = \sum\limits_C \left\{-\dfrac{w_{\mu\nu}}{2\pi}(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_a\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\right\}, \end{cases}$$

$$\text{(118)}\qquad \lim_{m=\infty} S_m^{(3)}(x)\cdot\begin{cases} = 0 & \text{when } x \neq a,\ x \neq b. \\ = \overline{K}_3\,F(a+0)\cdot & \text{when } x = a. \\ = \overline{\overline{K}}_3\,F(a+0)\cdot & \text{when } x = b. \end{cases}$$

It should be observed that the values of $\overline{K}_3$ and $\overline{\overline{K}}_3$ depend only upon the differential system whose characteristic functions form the terms of the expansion in question and are in particular entirely independent of the vector $\mathrm{F}(x)\cdot$

IV. $\mathrm{S}_m^{(4)}(x)\cdot$

Proceeding precisely as in the case of $S_m^{(3)}(x)\cdot$ we have upon defining $J_4\cdot$ by the relation

$$J_4 \cdot = Y(x)\,\Delta^{-1}\int_a^b W_b Y(b)\,(\delta^*_{ij})\,Z(t)\,R(t)\,F(t)\cdot dt,$$

$$S^{(4)}_m(x)\cdot = \frac{1}{2\pi i}\int_c J_4 \cdot d\lambda.$$

Further

$$\int_a^b W_b Y(b)\,(\delta^*_{ij})\,Z(t)\,R(t)\,F(t)\cdot dt =$$

$$\left(\sum_{h=1}^{n} w^{(b)}_{ih}\,\delta^*_{hh}\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}+B_h(b)-B_h(t)}\,\gamma_h(t)\,f_h(t)\,dt\right)$$

$$+\frac{1}{\lambda}\left(\sum_{h,k,p=1}^{n} w^{(b)}_{ih}\,\delta^*_{kk}\int_a^b e^{\lambda\{\Gamma_k(b)-\Gamma_k(t)\}+B_k(b)-B_k(t)}\,\gamma_p(t)\,f_p(t)\,[\delta_{hk}\psi_{kp}(t) + \delta_{kp}\,\varphi_{hk}(b)]\,dt\right)$$

$$=\frac{1}{\lambda}\left(\sum_{h=1}^{n} w^{(b)}_{ih}\,\delta^*_{hh}\left\{-f_h(b-0)+e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)+\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\frac{d}{dt}\,\{e^{B_h(b)-B_h(t)}f_h(t)\}\,dt\right\}\right)+\frac{(\varphi_2)}{\lambda}$$

and multiplying this matrix by the matrix $Y(x)\,\Delta^{-1}$, as given by formula (115) it follows that

$$J_4\cdot = \frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\overset{**}{\delta_{ii}}\}+B_i(x)-B_i(b)\overset{**}{\delta_{ii}}}\,\tau_{ik}(\lambda)\,w^{(b)}_{kh}\delta^*_{hh}\left\{-f_h(b-0)+e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)+\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\frac{d}{dt}\,\{f_h(t)e^{B_h(b)-B_h(t)}\}\,dt\right\}\right)+\frac{(\varphi_2)}{\lambda}.$$

The second matrix of the sum on the right, namely

$$\frac{1}{\lambda}\left(\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\delta_{ii}^{**}\}+B_i(x)-B_i(b)\delta_{ii}^{**}}\tau_{ik}(\lambda)\,w_{kh}^{(b)}\delta_{hh}^{*}e^{\lambda\Gamma_h(b)+B_h(b)}f_h(a+0)\right),$$

is seen to be of the type $(\varphi_2)/\lambda$, whereas the third, i.e.

$$\frac{1}{\lambda}\left[\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\delta_{ii}^{**}\}+B_i(x)-B_i(b)\delta_{ii}^{**}}\right.$$

$$\left.\tau_{ik}(\lambda)w_{kh}^{(b)}\,\delta_{hh}^{*}\int_a^b e^{\lambda\{\Gamma_h(b)-\Gamma_h(t)\}}\,\frac{d}{dt}\,\{f_h(t)\,e^{B_h(b)-B_h(t)}\}\,dt\right],$$

is also of this type, as is apparent from the fact that by lemma 3, each of the integrals with respect to t which actually occurs in any element of the matrix is of type φ_2, while the remaining factors are bounded.

The remaining matrix of importance in the expression for $J_4\cdot$, namely

$$\bar{J}_4(x)\cdot = \frac{-1}{\lambda}\left[\sum_{k,h=1}^{n} e^{\lambda\{\Gamma_i(x)-\Gamma_i(b)\delta_{ii}^{**}\}+B_i(x)-B_i(b)\delta_{ii}^{**}}\tau_{ik}(\lambda)w_{kh}^{(b)}\delta_{hh}^{*}f_h(b-0)\right],$$

again depends for its character upon the value of x, being of the type $(\varphi_2)/\lambda$ when and only when $x \neq a$ and $x \neq b$. By reasoning now familiar the results

$$\bar{J}_4(a)\cdot = \frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} \delta_{ii}^{*}\,\tau_{ik}(\lambda)\,w_{kh}^{(b)}\,\delta_{hh}^{*}f_h(b-0)+\varphi_2\right)$$

$$= \frac{-1}{\lambda}\,(\delta_{ij}^{*})\,\Omega^{-1}\,W_b\,(\delta_{ij}^{*})\,F(b-0)\cdot + \frac{1}{\lambda}\,(\varphi_2),$$

and

$$\bar{J}_4(b)\cdot = \frac{-1}{\lambda}\left(\sum_{k,h=1}^{n} \delta_{ii}^{**}\,\tau_{ik}(\lambda)\,w_{kh}^{(b)}\,\delta_{hh}^{*}f_h\,(b-0)+\varphi_2\right)$$

$$= \frac{-1}{\lambda}\,(\delta_{ij}^{**})\,\Omega^{-1}\,W_b\,(\delta_{ij}^{*})\,F(b-0)\cdot + \frac{1}{\lambda}\,(\varphi_2),$$

may be deduced, whereupon it follows that

$$S_m^{(4)}(x)\cdot\begin{cases} = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} (\varphi_2)\,\dfrac{d\lambda}{\lambda} & \text{when } x \neq a,\ x \neq b, \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} \{-(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\overset{-1}{\Omega}{}^{(\mu\nu)}\, W_b\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(b-0)\cdot + (\varphi_2)\}\,\dfrac{d\lambda}{\lambda} & \\ & \text{when } x = a. \\ = \sum\limits_C \dfrac{1}{2\pi i}\displaystyle\int_{C_{\mu\nu}} \{-(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\overset{-1}{\Omega}{}^{(\mu\nu)}\, W_b(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,F(b-0)\cdot + (\varphi_2)\}\,\dfrac{d\lambda}{\lambda} & \\ & \text{when } x = b. \end{cases}$$

Defining the matrices $\overline{K}_4$ and $\overline{\overline{K}}_4$ by the equations

$$(119)\qquad \begin{cases} \overline{K}_4 = \sum\limits_C \left\{ \dfrac{-w_{\mu\nu}}{2\pi}\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_b\,(\overset{*}{\delta}{}_{ij}^{(\mu\nu)}) \right\} \\ \overline{\overline{K}}_4 = \sum\limits_C \left\{ \dfrac{-w^{\mu\nu}}{2\pi}\,(\overset{**}{\delta}{}_{ij}^{(\mu\nu)})\,\overset{-1}{\Omega}{}^{(\mu\nu)}\,W_b\,(\delta_{ij}^{(\mu\nu)}) \right\}, \end{cases}$$

it follows, therefore, that

$$(120)\qquad \lim_{m\to\infty} S_m^{(4)}(x)\cdot \begin{cases} = 0 & \text{when } x \neq a,\ x \neq b \\ = \overline{K}_4\,F(b-0)\cdot & \text{when } x = a \\ = \overline{\overline{K}}_4\,F(b-0)\cdot & \text{when } x = b. \end{cases}$$

A summary of the various results as contained in formulas (110), (113), (118) and (120) is seen, in virtue of (105) and (111), to yield

$$\lim_{m\to\infty} S_m(x)\cdot = \tfrac{1}{2}\,F(x-0)\cdot + \tfrac{1}{2}\,F(x+0)\cdot \qquad \text{when } x \neq a,\ x \neq b,$$

$$\lim_{m\to\infty} S_m(a)\cdot = \tfrac{1}{2}\,F(a+0)\cdot + \overline{K}_3 F(a+0)\cdot + \overline{K}_4\,F(b-0)\cdot,$$

$$\lim_{m\to\infty} S_m(b)\cdot = \tfrac{1}{2}\,F(b-0)\cdot + \overline{\overline{K}}_3\,F(a+0)\cdot + \overline{\overline{K}}_4\,F(b-0)\cdot,$$

In consequence we have the following

THEOREM: Given that L is any vector differential system of the type

$$\overline{Y}'(x)\cdot = \{A(x)\lambda + B(x)\}\overline{Y}(x)\cdot,$$
$$\overline{W}_a\,\overline{Y}(a)\cdot + \overline{W}_b\,\overline{Y}(b)\cdot = 0,$$

which can be reduced by a change of the dependent variable to a system of type (46) for which (a), $R(x)$ and $B(x)$ are continuous together with their first derivatives (b), the functions $\gamma_j(x)$, $j = 1, 2, \ldots n$, satisfy the relations (96) and (97), and (c), the condition (ii) on page 89 is fulfilled. Then the development in characteristic functions of the system L which is associated with any vector $F(x)\cdot$ whose elements satisfy condition (98), converges to

$$\tfrac{1}{2}F(x-0)\cdot + \tfrac{1}{2}F(x+0)\cdot \quad \text{when } x \neq a,\ x \neq b,$$
$$H_aF(a+0)\cdot + J_aF(b-0)\cdot \quad \text{when } x = a,$$
$$H_bF(a+0)\cdot + J_bF(b-0)\cdot \quad \text{when } x = b,$$

the four matrices of constants H_a, J_a, H_b, and J_b being explicitly determined by the matrices $R(x)$, $A(x)$, $\overline{W}_a$ and $\overline{W}_b$ as stated.